# The Fundamental Theorem of Card Counting with Applications to Trente-et-Quarante and Baccarat [1])

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*Abstract:* The Fundamental Theorem of Card Counting is a unifying principle for the analysis of card games of chance which are characterized by sampling without replacement. The Theorem says (roughly) the "spread" in distribution of player expectations for partially depleted card packs increases with depletion of the card pack. Furthermore, average player expectation is non-decreasing (increasing under suitable hypotheses) with increasing depletion.

The Theorem is used to prove that significant favorable strategies based on card counting do not exist for Trente-et-Quarante or for "tie" bets in Nevada Baccarat. This is in sharp contrast with previous results for Blackjack and for Nevada Baccarat side bets on natural eight and natural nine.

## 1. Introduction

Games of chance which are repeated independent trials were well understood early in the development of probability theory. Recently probabilistic analysis has been extended to games of chance which are not repeated independent trials [EPSTEIN, 1967; THORP, 1961, 1962, 1969, unpublished manuscript; THORP and WALDEN, 1966, prebook]. A principal class has been those card games which feature sampling without replacement. Games in which more than one round is dealt from the card pack before reshuffling include blackjack, baccarat, trente-et-quarante, and poker.

A unifying principle which we call the Fundamental Theorem of Card Counting [THORP and WALDEN, 1966; pp. 316–7], has emerged from the separate analyses of such games. Consider for instance a game in which cards used on a round are put aside and successive rounds of play are dealt from an increasingly depleted pack. The cards are reshuffled before a round if the remaining unused cards would be insufficient to complete the round. Suppose there are two sides in the game, "the player" and "the house", and that best strategies and corresponding expectations can be calculated for each partially depleted pack of cards. Then the Fundamental Theorem says (in a manner which will be qualified and made precise) that the "spread" in the distribution of player expectations for partially depleted card packs increases with the depletion of the card pack. Furthermore

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the average player expectation is non-decreasing (even increasing under suitable hypotheses) with increasing depletion.

In particular, the Fundamental Theorem allows us to readily test for the existence of card counting strategies which yield positive expectation such as those which have been devised for blackjack and for some of the side bets in Nevada Baccarat. We shall prove the non-existence of practical favorable strategies for Trente et Quarante and for the "tie" bet in Nevada Baccarat. The method is to calculate the player best strategy and expectation for random subsets of the pack which are as small as might be expected to occur in actual play. Our samples will favor the player only slightly, and then infrequently. We then conclude (at the level of statistical significance of the sample of subsets) from the Fundamental Theorem that the opportunities are still less favorable for larger subsets and that therefore there are no practical favorable player strategies for Trente et Quarante or for the tie bet in Nevada Baccarat.

Certain previously unsuspected features of Trente et Quarante will be brought out in the analysis.

## 2. The Fundamental Theorem

We begin with the simplest form of the theorem, in which the player and the house each have a single fixed strategy. As an illustration and for motivation, let a pack of $N$ decks of (bridge) cards be shuffled. (*Shuffling* $n$ cards will mean that the $n!$ permutations are equally likely.) Suppose the house offers the wager "red at even money", meaning that the player can bet that the next card will be red. If the player so bets and the card is red, the player wins the amount of his bet. Otherwise he loses the bet.

A pack of $N$ decks of bridge cards has $26N$ red cards and $26N$ black cards, so the probability is one-half that the first card will be red. Hence the player has zero expectation and the bet is fair. If cards are put aside as used and play continues with the residual pack (sampling without replacement: an assumption we make from here on), the player who does not keep track of the used cards will still have zero expectation at each trial. But the player who keeps track of the used cards will, with his increased information, have an expectation that fluctuates around zero.

Let $k$ be the number of cards remaining in the pack. Let $F_k$ be the distribution function of player expectations given complete knowledge of the used cards. The Fundamental Theorem describes the behavior of $F_k$ as $k$ decreases.

In the general case we have a random variable $X_{S(k)}$ which is the amount the player receives for a one unit bet when a subset $S(k)$ of $k$ cards remain in the pack prior to the play. We assume that $X_{S(k)} = f(c_1, \dots, c_m)$ where $f$ is a function which characterizes the game, $k \geqq m$, and $\{c_1, \dots, c_m\}$ are the first $m$ cards, in order, of the subset $S(k)$. We assume (without loss of generality) that all cards are distinguishable, and let $(k)_m \equiv k(k-1)\cdots(k-m+1)$, $k \geqq m$. Then for the *particular subset* $S(k)$ of $k$ remaining cards $P(X_{S(k)} = x) = N[f(c_1,\ldots,c_m)]/(k)_m$ where $N[f(c_1,\ldots,c_m)]$ is the number of $\{c_1,\ldots,c_m\} \subset S(k)$ such that $f(c_1,\ldots,c_m) = x$. The random shuffling hypothesis yields $E(X_{S(k)}) = \{\sum f(c_1,\ldots,c_m) : c_1,\ldots,c_m \in S(k)\}/(k)_m$.

Let $F_k$, $f_k$, and $\mu_k$, respectively, be the distribution function, density function, and probability measure for $E(X_{S(k)})$, for fixed $k$. Thus they indicate the spectrum of opportunities which arise for the card counter when $k$ cards remain.

The $\mu_k$ for various $k$ (and hence the $f_k$ and $F_k$) will be connected by the concept of "convex contraction of a measure" which we now introduce.

Let $\mu$ be a real measure with finite support $\{x_1,x_2,\ldots,x_n\}$ and $\mu(x_i) = p_i$, $1 \leqq i \leqq n$. Definition. The process of forming the new measure $\nu$ defined by $\nu(x_i) = p_i - q_i$, $0 \leqq q_i \leqq p_i$, $1 \leqq i \leqq n$, and $\nu(x_{n+1}) = \sum q_i$, $x_{n+1} = q_i x_i/\sum q_i$, is called a *step*. We will also call the resulting measure a step when confusion is unlikely. If precisely $k$ of the $q_i$ are positive the process is a *k-step*. The point $x_{n+1}$ may be equal to one of the $x_i$, $1 \leqq i \leqq n$. The measure $\lambda$ is a *convex contraction* of the measure $\mu$ if $\lambda$ can be obtained from $\mu$ in a finite number of steps. If $\lambda$ is a convex contraction of $\mu$ then $\mu$ is a *convex dilation* of $\lambda$. We write $\lambda \leqq \mu$ or $\mu \geqq \lambda$. If $\lambda$ is a contraction of $\mu$ and $\lambda \neq \mu$, then $\lambda$ is a strict contraction of $\mu$ and we indicate this by $\lambda < \mu$.

Observe that the mean of a contraction $\lambda$ (dilation) is the same as that of the original measure $\mu$. Also contraction and dilation are transitive: $\mu_1 > \mu_2$ and $\mu_2 > \mu_3$ imply $\mu_1 > \mu_3$; $\lambda_1 < \lambda_2$ and $\lambda_2 < \lambda_3$ imply $\lambda_1 < \lambda_3$. The theory of convex contractions of measures is only given here for the case of finite support. It has been extended, however, to the general case.

The non-trivial step of least $k$ is a 2-step. The next lemma shows that any step may be obtained from a finite sequence of such simplest steps.

*Lemma 2.1:*

Any step may be obtained as a finite sequence of 2-steps.

*Proof:*

Suppose the lemma is proven for all $j$-steps where $2 \leqq j \leqq k$. The case $k = 2$ holds and by induction it suffices to establish it for an arbitrary $(k+1)$-step. Without loss of generality we may consider the measure $\mu(x_i) = p_i$ where $1 \leqq i \leqq n$ and $k+1 \leqq n$, and the step $\lambda(x_i) = p_i - q_i, 1 \leqq i \leqq k+1; \lambda(x_i) = p_i, k+2 \leqq i \leqq n$; and $\lambda(y) = \sum q_i$ where $y = q_i x_i/\sum q_i$.

Define the $k$-step $\lambda_k$ by $\lambda_k(x_i) = p_i - q_i$, $1 \leqq i \leqq k$; $\lambda_k(x_i) = p_i$, $k+1 \leqq i \leqq n$; and $z = \sum q_i x_i/\sum q_i$, $\lambda_k(z) = \sum q_i$, where the sums are for $1 \leqq i \leqq k$. We obtain $\lambda$ if we apply to this the 2-step $\lambda_2$ defined by $\lambda_2(x_i) = \lambda_k(x_i)$, $i \neq k+1$; $\lambda_2(x_{k+1}) = p_{k+1} - q_{k+1}$; and $\lambda_2(y) = \sum q_i$, $y = \left\{q_{k+1}x_{k+1} + z\left(\sum_{i \leqq k} q_i\right)\right\}\Big/\sum q_i = \sum x_i q_i/\sum q_i$ where sums are over $1 \leqq i \leqq k+1$ unless otherwise noted. This completes the proof.

By "contraction" we mean here any contraction which is obtained from a measure with finite support in a finite number of steps. In the general theory to be presented elsewhere these are called finitely generated contractions. From Lemma 2.1 we have at once the following characterization of such contractions.

*Corollary 2.2:*

A measure $\lambda$ is a convex contraction of the measure $\mu$ if and only if $\lambda$ can be obtained from $\mu$ in a finite sequence of 2-steps.

It is now easy to establish the following important property of contractions.

*Lemma 2.3:*

If $\lambda$ is a non-trivial convex contraction of $\mu$, then for each $\alpha > 1$ the $\alpha$th absolute moment of $\lambda$ about an arbitrary point $x_0$ is strictly less than the $\alpha$th absolute moment of $\mu$ about $x_0$. The first absolute moment of $\lambda$ about $x_0$ is less than or equal to that of $\mu$ about $x_0$.

*Proof:*

We find after some simplification that

$$\int |x - x_0|^\alpha d\mu - \int |x - x_0|^\alpha d\lambda$$
$$= a|x_1 - x_0|^\alpha + (1 - a)|x_2 - x_0|^\alpha - |a(x_1 - x_0) + (1 - a)(x_2 - x_0)|^\alpha$$

where $0 < a < 1$. By the strict convexity of $f(t) = t^\alpha, \alpha > 1$, this is strictly positive whenever $x_0$, $x_1$ and $x_2$ are distinct. The case $\alpha = 1$ is immediate.

*Question:*

Let $\mu$ and $\lambda$ be measures with finite support and suppose the conclusion of Lemma 2.3 holds. Is $\lambda < \mu$? If not, characterize for each $\mu$ all those $\lambda$ such that each corresponding $\alpha$th absolute moment, $\alpha > 1$, is strictly smaller for $\lambda$ than for $\mu$.

*Theorem 2.4:*

(Fundamental Theorem of card counting, first form: fixed strategies.) Given a game as above, the measures $\mu_k$ of the spectrum of player expectations satisfy $\mu_i \geqq \mu_j$ for all $n \geqq j > i \geqq m$. If $\mu_i$ is not constant, $\mu_i > \mu_j$. If $\mu_i$ is constant, $\mu_i = \mu_j$.

*Proof:*

Let $(n)_i = n(n-1)\cdots(n-i+1)$ and $\binom{n}{i} = (n)_i/i!$. Call an ordered list of $i$ cards an $i$-ordering, $(c_1, \ldots, c_i)$ and call a subset of $i$ cards an $i$-subset, $\{c_1, \ldots, c_i\}$.

The measure $\mu_i$ arises from the expectations of the $i$-subsets $S(i)$. In a well-shuffled $n$-card pack there are $\binom{n}{i}$ distinct equiprobable $i$-subsets, each with probability $\binom{n}{i}^{-1}$. Each $S(i)$ has $i!$ distinct orderings, and over all $i$-subsets there are $(n)_i$ distinct $i$-orderings in all.

We use these $i$-orderings to construct $\mu_j$ as a contraction of $\mu_i$. Let $(c_1, \ldots, c_i)$ be a typical $i$-ordering. Assign $(n)_i/(n)_j = 1/(n-i)_{j-i}$ of the probability of $(c_1, \ldots, c_i)$ to each of the $(n-i)_{j-i}$ $j$-orderings $(c_1, \ldots, c_i, x_{i+1}, \ldots, x_j)$. This assignment, when taken over all $i$-orderings, includes each $j$-ordering exactly once and uses all the probability of all the $i$-orderings.

Each $(c_1,\ldots,c_i)$ is contained in the corresponding $i$-subset $\{c_1,\ldots,c_i\}$ and each $(c_1,\ldots,c_i,x_{i+1},\ldots,x_j)$ is in $\{c_1,\ldots,c_i,x_{i+1},\ldots,x_j\}$. The assignment takes $1/(n-i)_{j-i}$ of the probability of each $(c_{\sigma(1)},\ldots,c_{\sigma(i)})$ in $\{c_1,\ldots,c_i\}$ into a $j$-ordering of $\{c_1,\ldots,c_i,x_{i+1},\ldots,x_j\}$. Hence $\{c_1,\ldots,c_i\}$ contributes $(j-i)!/((n-i)_{j-i}\binom{n}{i}) = i!(j-i)!/(n)_j$ to $\{c_1,\ldots,c_i,x_{i+1},\ldots,x_j\}$.

But contributions to the given $j$-subset come from $\binom{j}{i}$ $i$-subsets for a total contribution to $\{c_1,\ldots,c_i,x_{i+1},\ldots,x_j\}$ of $\binom{n}{j}^{-1}$, which is correct. Let the expectation of the $i$-subset $\{c_1,\ldots,c_i\}$ be denoted $E\{c_1,\ldots,c_i\}$. Suppose a given $j$-subset $\{d_1,\ldots,d_j\}$ contains the $i$-subset $\{c_1,\ldots,c_i\}$. We will show that $E\{d_1,\ldots,d_j\} = \frac{1}{N}\sum E\{c_1,\ldots,c_i\}$ where the sum is over all $i$-subsets of $\{d_1,\ldots,d_j\}$ and $N$ is the number of such $i$-subsets, namely $N=\binom{j}{i}$.

Each $i$-subset $\{c_1,\ldots,c_i\}$ has $i!$ orderings $(c_{\sigma(1)},\ldots,c_{\sigma(i)})$ and $E\{c_1,\ldots,c_i\} = \frac{1}{i!}\sum_\sigma f(c_{\sigma(1)},\ldots,c_{\sigma(m)})$, where $\sigma$ is a permutation of $1,\ldots,i$. Similarly $E\{d_1,\ldots,d_j\} = \frac{1}{j!}\sum_\tau f(d_{\tau(1)},\ldots,d_{\tau(m)})$, where $\tau$ is a permutation of $1,\ldots,j$.

Now

$$\begin{aligned}\frac{1}{N}\sum E\{c_1,\ldots,c_i\} &= \frac{1}{N}\sum\frac{1}{i!}\sum_\sigma f(c_{\sigma(1)},\ldots,c_{\sigma(m)})\\ &= \frac{1}{i!N}\frac{1}{(j-i)!}\sum_\tau f(d_{\tau(1)},\ldots,d_{\tau(m)}) = \frac{1}{j!}\sum_\tau f(d_{\tau(1)},\ldots,d_{\tau(m)})\\ &= E\{d_1,\ldots,d_j\}.\end{aligned}$$

The probability of each $i$-subset is $\binom{n}{i}^{-1}$. The part assigned to a given $j$-subset from a given $i$-subset is $\lambda = i!(j-i)!/(n)_j$. Thus for $E\{d_1,\ldots,d_j\}$ to arise from a step from $\mu_i$ by the above assignment we require $\sum E\{c_1,\ldots,c_i\}\cdot\lambda/\sum\lambda = \sum E\{c_1,\ldots,c_i\}/N$, as was shown above.

Now we note that all the probability of each $S(i)$ is precisely used so the total result of performing all the steps of the assignment (in any order) not only yields $\mu_j$ but is consistent. This follows from the observation that the number of $j$-subsets containing a given $i$-subset is $\binom{n-i}{j-i}$ and since the $i$-subset contributes $i!(j-i)!/(n)_j$ to each of these, the total probability required from the $i$-subset is $\binom{n}{i}^{-1}$, which is precisely what is available. This shows that $\mu_j \leqq \mu_i$.

Since any contraction of a constant measure is the same constant measure, if $\mu_i$ is constant then $\mu_j = \mu_i$ for $n \geqq j > i$.

Next suppose that $\mu_i$ is not constant. Then there are two $i$-subsets $\{c_1,\ldots,c_i\}$ and $\{d_1,\ldots,d_i\}$ with $E\{c_1,\ldots,c_i\} \neq E\{d_1,\ldots,d_i\}$. Then in the sequence $E\{c_1,\ldots,c_i\}$, $E\{d_1,c_2,\ldots,c_i\}$, $E\{d_1,d_2,c_3,\ldots,c_i\},\ldots,$ $E\{d_1,\ldots,d_{i-1},c_i\}$, $E\{d_1,\ldots,d_i\}$, two consecutive numbers must be unequal. Suppose $E\{d_1,\ldots,d_k,c_{k+1},\ldots,c_i\} \neq$

$E\{d_1, \ldots, d_{k+1}, c_{k+2}, \ldots, c_i\}$. Then there is a $j$-subset which contains $\{d_1, \ldots, d_{k+1}, c_{k+1}, \ldots, c_i\}$ and each such $j$-subset arises (by the assignment of the proof) from a nontrivial step. Therefore $\mu_j < \mu_i$.

### 3. Application to Trente et Quarante

Trente et Quarante is played with six packs of cards shuffled together, forming a total deck of 312 cards. Aces count 1, Jacks, Queens, and Kings count 10, and the other card are counted equal to their rank. A dealer representing the "house" deals two rows of cards, the first corresponding to Black and the second to Red. Cards are dealt in the first row until the point total is 31 or more. This is repeated for the second row. Thus Red and Black each are associated with numbers ranging between 31 and 40.

A one unit bet on Black (Red) wins one additional unit if the total of the first row is less than (greater than) that of the second and is lost if the total is greater (less). If the totals of the two rows are equal and greater than 31 there is a tie and all bets are refunded. If both rows total 31, the house wins one half of the bets (unless they have been "insured").

Color is determined by the first card of the first row. If Red (Black) wins and the first card is red (black) then Color wins and Inverse loses. If Red (Black) loses and the first card is red (black), then Color loses and Inverse wins. Color and Inverse tie if both totals are equal and between 32 and 40. They lose one half the bet if both totals are 31.

All bets may be insured by wagering an additional 1%. It cancels the loss if both rows total 31, is unaffected by other ties, and is lost to the house when the two rows have unequal totals.

Probabilities for the totals 31 through 40 were first computed by POISSON [1825] and subsequently by JOSEPH BERTRAND with the assumption of sampling with replacement, equivalent to an infinite deck.

Table 1. Exact probability a row terminates in $N$ cards, compared with the infinite deck approximation

| Cards required | Probabilities | |
|---|---|---|
| | Exact | $\infty$-deck |
| 4 | .26082 | .17453 |
| 5 | .36705 | .32761 |
| 6 | .23962 | .27778 |
| 7 | .09688 | .14355 |
| 8 | .02804 | .05576 |
| >8 | .00759 | .02086 |
| Total | 1.00000 | 1.00009 |

Numbers based on the infinite deck approximation may be in considerable error. Table 1 compares the exact probability, obtained with a computer, that a row will end in $N$ cards, with the infinite deck approximation given in [BOLL, 1936], page 200.

An exact calculation was made of the probabilities $P(T|N)$ of various totals $T$ given that the row terminates in $N$ cards. The results appear as Table 2. Note that for $N = 4$, 5 and 6, $P(T|N)$ decreases as $T$ increases. However $P(T|8)$ increases as $T$ increases, to a maximum at $T = 36$, then decreases. This suggests for $N = 9$ a maximum at $T \geqq 36$ and for $N \geqq 9$ (cumulative) a maximum at $T \geqq 36$, perhaps at $T = 40$, with $P(T|N \geqq 9)$ increasing as $T$ increases.

Let $\Delta_N \equiv \max\{P(T|N): 31 \leqq T \leqq 40\} - \min\{P(T|N): 31 \leqq T \leqq 40\}$ and note that $\Delta_4 = 0.0450058$, $\Delta_5 = 0.0357255$, $\Delta_6 = 0.0131792$, $\Delta_7 = 0.0026006$, and $\Delta_8 = 0.0002581$. It seems plausible that $\Delta_9 + \Delta_{10} + \cdots < 0.000025$ whence

$$\left|0.00075863 + \sum_{N=4}^{8} P(T|N) - P(T)\right| < .000025 \text{ and probably less than } .000015.$$

Table 2. The exact probability $P(T|N)$ of a total $T$ given that the row terminates in $N$ cards

| $T$ | $N = 4$ | $N = 5$ | $N = 6$ | $N = 7$ | $N = 8$ | $N > 8$ |
|---|---|---|---|---|---|---|
| 31 | 0535835 | 0521411 | 0285618 | 0103221 | 0027563 | |
| 32 | 0454004 | 0503734 | 0281976 | 0103554 | 0027917 | |
| 33 | 0382147 | 0477568 | 0277042 | 0103546 | 0028242 | |
| 34 | 0315609 | 0444029 | 0270034 | 0103093 | 0028518 | |
| 35 | 0258280 | 0404907 | 0260055 | 0102044 | 0028713 | |
| 36 | 0205556 | 0361045 | 0246739 | 0100124 | 0028787 | |
| 37 | 0161382 | 0314125 | 0229517 | 0097059 | 0028677 | |
| 38 | 0121041 | 0264846 | 0208373 | 0092492 | 0028294 | |
| 39 | 0088544 | 0214676 | 0183062 | 0086107 | 0027519 | |
| 40 | 0085777 | 0164156 | 0153826 | 0077548 | 0026206 | |
| $P(\text{Total}\|N)$ | 2608174 | 3670497 | 2396242 | 0968788 | 0280436 | 0075863 |

Table 3 compares estimated values for the exact probability $P(T)$ of a given total and the infinite deck approximation.

The greatest absolute errors in the infinite deck approximation are probably no more than $(290 + 15)10^{-6}$ or about $3 \times 10^{-4}$ for $P(40)$. The greatest percentage error, also for $P(40)$, is probably less than 0.6%.

The probability that both rows total 31 would be $.148061^2 = 2.1922\%$ by the infinite deck approximation and $.148123^2 = 2.1940\%$ from our estimate. These calculations both neglect the fact that if one row totals 31 the conditional probability that the other row then totals 31 may be different. The probability that both rows are equal and exceed 31 is given by BOLL [1936] in the infinite-deck case as 8.78254%. By comparison Table 3 yields 8.78425%.

Table 3. Our estimate ($\pm$ .000015) of the exact probabilities

| $T$ | $W$ Estimate of exact $P(T)$ | $B$ Infinite deck approximation | $(B - W)\,10^6$ |
|---|---|---|---|
| 31 | 0.148123 | 0.148061 | − 62 |
| 32 | 0.137877 | 0.137905 | 28 |
| 33 | 0.127613 | 0.127513 | −100 |
| 34 | 0.116886 | 0.116891 | 5 |
| 35 | 0.106158 | 0.106050 | −108 |
| 36 | 0.094983 | 0.094968 | 15 |
| 37 | 0.083834 | 0.083750 | − 84 |
| 38 | 0.072263 | 0.072317 | 54 |
| 39 | 0.060749 | 0.060716 | − 33 |
| 40 | 0.051509 | 0.051799 | 290 |

Let $B$ be the event black wins, $R$ the event red wins, $t = 31$ the event that there is a tie at 31, and $t > 31$ the event that there is a tie greater than 31. Then $P(B) + P(R) + P(t = 31) + P(t > 31) = 1$ and symmetry arguments show that $P(B) = P(R)$ for a randomly shuffled subset of cards:

*Lemma 3.1:*
$P(R) = P(B)$ hence $E(R) = E(B) = -\frac{1}{2}P(t = 31)$.

*Proof:*
Let $\pi = \{(x_1, \ldots, x_k; y_1, \ldots, y_l); z_1, \ldots, z_m\}$ be a permutation of the deck such that $(x_1, \ldots, x_k; y_1, \ldots, y_l)$ is the coup, and such that red wins.

Then, under random shuffling, $\emptyset(\pi) = \{(y_1, \ldots, y_l; x_1, \ldots, x_k); z_1, \ldots, z_m\}$ is a distinct and equiprobable permutation of the deck in which black wins.

Note that $\emptyset$ is a $1 - 1$ onto correspondence between those deck permutations which give a red win and those which give a black win. Thus they are numerically equal. Under random shuffling, all permutations are equiprobable so $P(R) = P(B)$.

For a complete well shuffled pack, it follows that the expectation $E(B)$ of a bet on black to win *on the next deal* is

$$E(B) = P(B) - P(R) - .5P(t = 31) = .5P(t = 31) \doteq -1.0970\%.$$

If $t > 31$ is ignored, i.e., if the bet remains and play is continued until some money is won or lost, then we find

$$E(B) = -.5P(t = 31)/(1 - P(t > 31)) \doteq -1.2026\%.$$

If the bettor takes insurance, the expectation $E(B_i)$ of black is $-.01(P(B) + P(R)) = -0.8902\%$. If $P(t > 31)$ is ignored, this becomes $-0.9759$. However, with insurance the bets are also unaffected when $t = 31$ occurs. Ignoring this outcome as well yields precisely $E(B_i) = -1\%$. The reasons for considering alternate figures for the expectation, ignoring ties, are discussed in [THORP, 1973].

These figures are all the same for bets on red, color, and inverse.

When the pack is partially used, $P(t = 31)$ may change. For instance, if only cards remain whose values are multiples of 2, $P(t = 31) = 0$ hence $E(B) = E(R) = 0$. However, because $P(B) = P(R)$ always, we have $E(B) = E(R) = -.5P(t = 31) \leqq 0$. Similarly $E(B_i) = E(R_i) = .01(P(B) + P(R)) \leqq 0$. Equality occurs only when the remaining cards all have rank 10. Thus no card counting strategy can give positive expectation for bets on red or black.

It has been assumed in all previous studies of the game that the same was true for bets on color and inverse. This is true for a full pack, as Lemma 3.2 shows.

*Lemma 3.2:*

If each rank has an equal number of red and black cards, then $P(C) = P(I)$, hence $E(C) = E(I) = -\frac{1}{2}P(31)$. (We neglect insurance and other variations for simplicity; the main conclusion, that $P(C) = P(I) \leqq 0$ is unaffected, anyhow.)

*Proof:*

Let $\pi = \{(x_1,\ldots,x_k;y_1,\ldots,y_l);z_1,\ldots,z_m\}$ be a permutation of the deck such that $(x_1,\ldots,x_k;y_1,\ldots,y_l)$ is the coup, and such that color wins. Define $\emptyset$ by $\emptyset(\pi) = \{(\bar{x}_1,\ldots,\bar{x}_k;\bar{y}_1,\ldots,\bar{y}_l);\bar{z}_1,\ldots,\bar{z}_m\}$, where $c \to \bar{c}$ is a one-to-one mapping of each rank onto itself which reverses colors. Then $\emptyset(\pi)$ is a permutation such that the same row wins, and hence such that inverse wins. Note that $\emptyset$ is $1-1$ onto between permutations where color wins and those where color loses. The conclusion follows. Note too that $\emptyset(\pi) = \{(\bar{x}_1,\ldots,\bar{y}_l);z_1,\ldots,z_m\}$ or $\emptyset(\pi) = \{(\bar{x}_1,x_2,\ldots,y_l);z_1,\ldots,z_m\}$ are not in general adequate.

However, the next example shows that the conclusion of Lemma 3.2 is false in general.

*Example:*

A randomly shuffled subset with positive expectation for inverse: $E(I) = 25\%$.

Suppose the cards remaining to be played consist of seven black tens and one red five. Then inverse wins if the red five is in the pack in locations 1, 5, 6, 7, or 8 and color wins if the red five is in locations 2, 3, or 4. Thus $P(I) = 5/8$, $P(C) = 3/8$, and $E(I) = 25\%$.

The fact that color and inverse differ in this way from red and black seems to have been previously unsuspected.

The example shows that $E(I)$ and $E(C)$ may vary with deck composition. Whether these variations are of sufficient magnitude and frequency can be determined with the aid of Theorem 2.4. Since that theorem shows the effect increases as the number $k$ of remaining cards decreases, we chose the smallest value, $k = 20$, that seemed fairly likely to arise in practice. A computer was used to simulate this procedure: A subset of size 20 was randomly chosen from the full pack, then shuffled and dealt 40,000 times. The frequently of $B, R, C, I, t = 31$, and $t > 31$ was tallied and from this observed expectations were calculated. For $E(B)$ the standard deviation of observations from the true value is approximately $0.5\%$. Thirty-two observations were made. A summary of results appears in Table 3.1. The sample standard deviations for $P(C)$ and $P(I)$ are somewhat higher than for $P(R)$ and $P(B)$. We take this as evidence that the distribution of $P(C)$ and $P(I)$ spreads somewhat more at $k = 20$.

Table 3.1. A computer simulation of 40,000 deals from each of 32 randomly selected subsets

| | $P(B)$ | $P(R)$ | $P(C)$ | $P(I)$ | $P(t = 31)$ | $P(t > 31)$ | | |
|---|---|---|---|---|---|---|---|---|
| Sample mean | .44587 | .44697 | .44556 | .44730 | .02054 | .08661 | | |
| Sample $\sigma$ | .00588 | .00428 | .00624 | .00769 | .00696 | .00512 | | |
| | $E(B)$ | $E(R)$ | $E(B_i)$ | $E(R_i)$ | $E(C)$ | $E(I)$ | $E(C_i)$ | $E(I_i)$ |
| Maximum | $+0.6\%$ | $+0.6\%$ | $+0.8\%$ | $+0.6\%$ | $+0.6\%$ | $+2.7\%$ | $+0.5\%$ | $+2.3\%$ |
| Number positive | 2 | 2 | 1 | 3 | 5 | 6 | 4 | 5 |

For each of the 32 observations we then calculated the observed expectations of the four bets without and with insurance. The subscript "$i$" denotes insurance. This display of the data shows that there were more instances of positive expectation for $C$, $I$, $C_i$, and $I_i$ than for $B$, $R$, $B_i$, and $R_i$, and that the maxima were greater. However, the effect for $C$, $I$, $C_i$, and $I_i$ is so small that it seems statistically very unlikely that for $k = 20$ the effect is of practical importance to the card counter. The effect is undetectable by a similar analysis for $k = 30$.

Since Theorem 2.4 shows that the effect is still less for greater $k$, the conclusion is that there are no card counting strategies for the color and inverse bets at Trente et Quarante which yield a practically important player advantage.

## 4. Application to the Nevada Baccarat Tie Bet

In a previous paper we discussed Nevada Baccarat at length [THORP and WALDEN, 1966]. We showed how statistical analysis led to a favorable card counting strategy for the side bets on natural eights and nines. Those bets have since disappeared and a new bet on "ties" has appeared. In the event the Banker and player hands have the same total (a "tie"), this bet gains nine times the amount bet. Otherwise the bet is lost. The probability of a tie has been shown in [THORP and WALDEN, 1966] to be $9.5156\%$, hence the expectation $E(t)$ of the bet is $-4.8440\%$. However, it is clear that $P(t)$, and thus $E(t)$, depend on the subset of unplayed cards. For instance, in the extreme and improbable event that the residual deck consists only of ten value cards, $P(t) = 1$ and $E(t) = 9$. Thus card counting strategies are potentially advantageous.

To determine whether card counting strategies are practical, we again applied the Fundamental Theorem. By computer simulation random subsets of size $N$ were selected from a complete 416 card pack. The values of $N$ chosen were 208, 156, 104, 52, 26, and 13. One-hundred subsets were chosen for each $N$ except $N = 13$, where 154 were used. The value of $P(t)$ was calculated for each subset and then the standard deviation $\sigma_N$ was calculated for the sample for each $N$. The results are summarized in Table 4.1. The values for $\sigma_N$ are described approximately by the formula $\sigma_N = .114N^{-.84}$. Using the formula as a rough estimator, zero expectation occurs at the $2\sigma_N$ level of significance at about $N = 97$. It occurs at the $\sigma_N$ level of significance at about $N = 43$, which is also where advantages of 4.84% or more occur at the $2\sigma_N$ level. Thus the advantages which occur with complete knowledge of the used cards are limited to the extreme end of the pack and are generally not large.

Table 4

| Subset size $N$ | Number of observations | $\sigma_N$ | $P(t) + \sigma_N$ | Corresponding advantage | $P(t) + 2\sigma_N$ | Corresponding advantage |
|---|---|---|---|---|---|---|
| 208 | 100 | .00140 | .09656 | −3.44 | .09796 | − 2.04 |
| 156 | 100 | .00176 | .09691 | −3.08 | .09868 | − 1.32 |
| 104 | 100 | .00233 | .09749 | −2.51 | .09981 | − 0.18 |
| 52 | 100 | .00400 | .09916 | − .84 | .10316 | + 3.16 |
| 26 | 100 | .00645 | .10161 | +1.61 | .10806 | + 8.06 |
| 13 | 154 | .01475 | .10991 | +9.91 | .12466 | +24.66 |

We conclude that practical card counting strategies are at best marginal, and at best precarious, for they are easily eliminated by shuffling the pack at, say, $N = 26$.

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