THE RELATION BETWEEN A COMPACT LINEAR OPERATOR AND ITS CONJUGATE

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1. Introduction. We present a systematic account of known theorems relating compact linear operators and their conjugates. Examples are given showing that all the theorems which are "possible" in a certain broad sense are already known. The general method is that of [6].

In what follows, X and Y are normed linear spaces. $[X, Y]$ is the set of bounded linear operators with domain X and range in Y. T denotes a linear operator and $R(T)$ is its range. T is compact if, for each bounded sequence (x_n) in X, (Tx_n) has a convergent subsequence. $[X, Y]_c$ stands for the set of compact linear operators with domain X and range in Y. We say that T has an inverse if $Tx=0$ implies $x=0$, *i.e.*, if T sets up a 1-1 mapping of X onto $R(T)$. The inverse mapping T^{-1} is also linear. X' is the space of bounded linear functionals on X, normed in the usual way. If $T\in[X, Y]$, the operator T' is defined as follows: $T'y'=x'$, where $x'\in X'$ is defined by $x'(x)=y'(Tx)$, all $x\in X$. $T'\in[Y', X']$ and $\|T'\|=\|T\|$.

Motivated by the known theorems relating a bounded operator T and its conjugate, we classify various possibilities for T by:

I: $R(T)=Y$ (indicated by writing $T\in I$).

II: $R(T)\neq Y$ but $\overline{R(T)}=Y$ (written $T\in\text{II}$, and so on for succeeding cases).

III: $\overline{R(T)}\neq Y$.

1: T^{-1} exists and is bounded.

2: T^{-1} exists but is not bounded.

3: T^{-1} does not exist.

If $T\in\text{II}$ and $T\in 1$, we combine this by writing $T\in\text{II}_1$. Thus there are nine possibilities for T. Similarly, T' has nine classifications. Thus, the pair (T, T') has 81 classifications. We call these 81 classifications the states of the pair (T, T'). If, for example, $T\in\text{II}_1$ and $T'\in\text{III}_3$, we say that the pair is in state $(\text{II}_1, \text{III}_3)$.

Taylor and Halberg [6] found the possible states for the pair (T, T') when $T\in[X, Y]$. They organized their results schematically as shown in Figure 1. Referring to this figure, if a box is crossed out, this means the corresponding state is impossible for any pair (T, T'), regardless of the choice of X and Y. If, on the other hand, a box is not crossed out, it is "possible." Some "possible" states become impossible when X and Y are suitably restricted. This is symbolically noted in the square representing the state. The key below the figure gives the meanings of the symbols.

Remark. A generalization to unbounded operators of the Taylor-Halberg state diagram for bounded operators has been made by Goldberg [4]. The details are to appear in the Pacific Journal of Mathematics and the results are:

1) If $T: X \to Y$ is an operator whose graph is closed in the product topology of $X \times Y$ and the domain of T is dense in X, then the Taylor-Halberg state diagram remains true. (It is remarkable that such a vast extension of the operator class does not result in the opening up of a single new square in the state diagram.)

2) If $T: X \to Y$ is *any* linear operator with dense domain (Linear is being used to mean $T(ax+by) = aTx + bTy$; there are no topological implications.), the state diagram is the same as the Taylor-Halberg state diagram except that all the X and Y_r symbols are to be erased.

State Diagram for Bounded Linear Operators ($[X, Y]$)

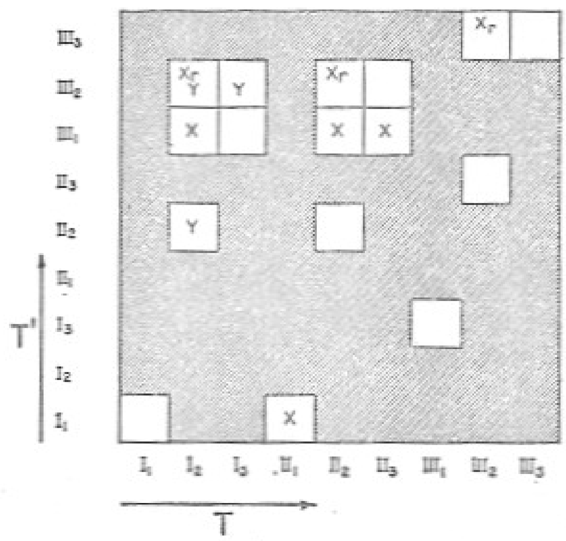

FIG. 1

Key: X: cannot occur if X is complete; Y: cannot occur if Y is complete; X_r: cannot occur if X is reflexive.

2. Derivation of the state diagram for compact operators. An operator is bounded if and only if it sends bounded sequences into bounded sequences. Convergent sequences are bounded. Therefore every compact operator is a bounded operator so $[X, Y]_c$ is a subset of $[X, Y]$. Hence the state diagram for $[X, Y]_c$ can be thought of as a restriction (fewer open squares) of the diagram for $[X, Y]$. The restriction of the state diagram for $[X, Y]$ to the state diagram for $[X, Y]_c$ follows from two lemmas below. Because of the form of Lemma 1, the state diagram will be established under the assumption that X and Y are infinite dimensional. The simple case in which X and/or Y is finite dimensional is considered separately.

LEMMA 1. *If T is compact and $T \in 1$, then* dim $R(T)$ *and* dim X *are finite and equal* ([5], p. 115).

From Lemma 1, if X is infinite dimensional and T is compact, state 1 is impossible for T. This means that the first, fourth, and seventh columns of the $[X, Y]$ state diagram correspond to impossible states in the $[X, Y]_c$ state diagram. Similarly, Y' is infinite dimensional as a consequence of our assumption that Y is infinite dimensional, so the first, fourth and seventh rows are impossible states in the $[X, Y]_c$ state diagram. Thus we see from Lemma 1 that seven of the sixteen possible states for $[X, Y]$ are impossible for $[X, Y]_c$. To see the full strength of Lemma 1, observe that forty-five squares, more than half the total, are shown to be impossible by Lemma 1, without invoking any other theorems.

State Diagram for Compact Linear Operators ($[X, Y]_c$)

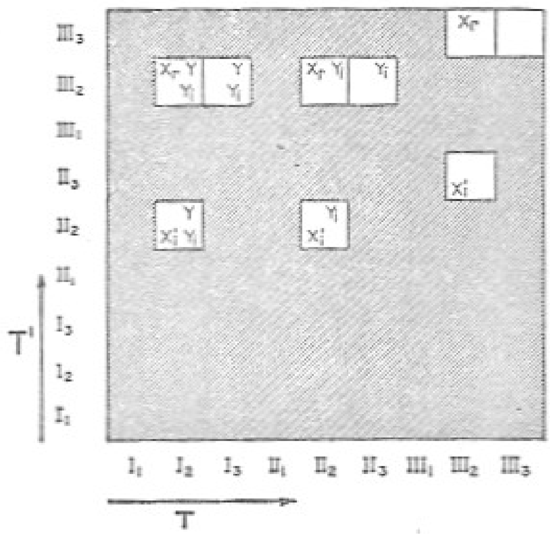

FIG. 2

Key: Y: impossible if Y is complete; X_r: impossible if X is reflexive; $Y_i(X_i')$: impossible if $Y(X')$ is inseparable.

The following result of Banach shows that the state diagram for compact operators is further restricted for certain choices of X' or Y.

LEMMA 2. *If T is compact, then $R(T)$ is separable* ([1], p. 96).

If T is not in III and Y is inseparable, then $\overline{R(T)}$ is inseparable. Hence $R(T)$ is inseparable and Lemma 2 shows that T is not compact. Thus, if T is compact, states I and II are impossible and the first six columns are deleted.

Similarly, if X' is inseparable the first six rows are impossible states. The resultant state diagram is shown in Figure 2. The examples of Section 3 show that this is the final form.

Remark. In constructing the state diagram for X and/or Y finite dimensional, it is simplest to consider in turn each of the three cases listed below. The results given are immediate. If neither X nor Y is (0), the two states shown in each case always exist. No others exist.

1. $\dim X = \dim Y$; X and Y are both finite dimensional. (III_3, III_3), (I_1, I_1).
2. $\dim X$ is greater than $\dim Y$; Y is finite dimensional. (III_3, III_3), (I_3, III_1).
3. $\dim Y$ is greater than $\dim X$; X is finite dimensional. (III_3, III_3), (III_1, I_3).

Obviously II and 2 are impossible for T and T' so the state diagram (Fig. 3) is drawn with only the 16 squares listed.

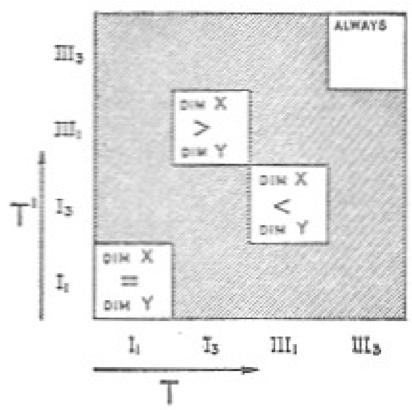

Fig. 3

Key: The four open squares correspond to existing states if and only if the conditions within the open squares are fulfilled.

3. All the states shown as possible exist "maximally." The examples to follow show that every state shown as "possible" actually exists. The examples are "maximal" in the sense that (1) X and Y are each complete, or even reflexive, and (2) X' and Y are inseparable, unless this is already forbidden by the $[X, Y]_c$ state diagram. Making X complete and Y reflexive is suggested by the occurrence of these restrictions in the $[X, Y]$ state diagram, for they do not appear as restrictions in the $[X, Y]_c$ state diagram.

(III_3, III_3): All the operators with finite-dimensional ranges are in this state for any infinite-dimensional (X, Y) pair. An (X, Y) pair such that X and Y are reflexive, and X' and Y are inseparable, is: $X = Y = l^2(Q)$ where the cardinality of Q is greater than $\aleph_0$. Note: For any set Q, $l^2(Q)$ is defined as the set of those scalar-valued functions with domain Q such that (1) at most a countable number of the coordinates are nonzero and (2), $\sum |x_q|^2$ is finite. x_q is the qth coordinate of a typical function x. The norm of x is the square root of the sum in (2).

(II_2, II_2): $X = Y = l^2$; let (u_k), $k=1, 2, \cdots$ be a countable orthonormal basis in l^2. Define T by $Tu_k = 2^{1-k}u_k$. T is compact because it satisfies the criterion (see, *e.g.*, [3], Th. 7, Cor.) $\sum |t_{ij}|^2$ is finite, where (t_{ij}) is the matrix corresponding to T. $T' = T$; hence the state must be, according to the diagram, (II_2, II_2) or (III_2, III_2). But $\overline{R(T)} = Y$ because every element with at most a finite number of nonzero coordinates is in $R(T)$.

(II_3, III_2): $X = Y = l^2$; define T by $Tu_k = 2^{1-k}u_{k-1}$, $k=2, 3, \cdots$ and $Tu_1 = 0$. The arguments of the preceding example can be used to show T is compact and in (II_3, III_2). To make this example "maximal," we must modify it so that X is inseparable. Let X_0 be the direct sum of l^2 and $l^2(Q)$, where $l^2(Q)$ is a nonseparable Hilbert space. Define T_0 as the direct sum of T_1 and T by setting $T_1 = 0$ on $l^2(Q)$. This T_0, with the pair (X_0, Y), is the desired example.

This device can be used to make X' or Y inseparable in all the following state examples, unless the state diagram already forbids this.

(III_2, II_3): Use the conjugate of the operator in the preceding example.

(II_2, III_2): $X = l^1$, $Y = l^2$; define T by $Tu_k = 2^{1-k}u_k$, as in example (II_2, II_2). It is shown in [6], p. 104 that the state is (II_2, III_2). Compactness is shown as follows: Let T_0 be the same as T except that the domain of T_0 is l^2. Let I_0 be the canonical imbedding of l^1 in l^2 defined by $I_0x = x$. It is readily verified that $\|I_0\| = 1$. Notice that $T = T_0I_0$ and that T_0 is compact by the previously used criterion. Hence T is compact.

(III_2, III_2): $X = l^1$, $Y = l^2$; $Tu_k = 2^{-k}u_{k+1}$, $k=1, 2, \cdots$; T is shown to be compact by the method of the previous example.

(I_2, II_2), (I_2, III_2), (I_3, III_2): Examples of these states are obtained by modifying the previously given examples of the states (II_2, II_2), (II_2, III_2), (II_3, III_2), respectively, using the procedure given in [6].

4. The state diagram for weakly compact operators is the same as that for bounded operators. The weak topology on X is the weakest (coarsest, smallest) topology making every element of X' continuous. A linear operator $T \in [X, Y]$ is weakly compact if the weak closure of $T(S_X)$ is compact in the weak topology on Y (S_X is the unit sphere in X). The set of weakly compact operators is designated by $[X, Y]_{wc}$. Observe that $[X, Y]_c$ is a subset of $[X, Y]_{wc}$ and that $[X, Y]_{wc}$ is itself a subset of $[X, Y]$. To show the second inclusion, note that a weakly compact set is weakly bounded hence, by the uniform boundedness principle it is norm-bounded. The first inclusion is true because $T \in [X, Y]_c$ means that the norm closure of $T(S_X)$ is norm-compact and hence weakly compact. Now note that the norm closure of a manifold equals the weak closure, therefore the weak closure of $T(S_X)$ is weakly compact.

We now show the equality of state diagrams for $[X, Y]$ and $[X, Y]_{wc}$. We need the following result:

THEOREM. *Y is norm reflexive if and only if S_Y is weakly compact* (see, e.g., [2], Ch. IV, Sec. 5, n° 2, Prop. 6).

This theorem shows us that $[X, Y]=[X, Y]_{wc}$ if Y is reflexive. Thus the examples in [6] which show a state is possible and have Y reflexive also show the state is possible for $[X, Y]_{wc}$. The only possible states in the $[X, Y]$ diagram not included by this are (I_2, II_2), (I_2, III_2), (I_3, III_2). But if T is compact, T is weakly compact so the examples of these three states given in the $[X, Y]_c$ diagram suffice for the $[X, Y]_{wc}$ diagram.

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Remark on: "The Relation Between a Compact Linear Operator and its Conjugate", E. O. Thorp, Am. Math. Monthly, Vol. 66 No. 9, Nov. 1959.

At the end of §3 it is asserted that reducing the range space to the range of the operator leaves the operator in question compact. This is not obvious, but is justified, as the following discussion shows.

If $T: X \to Y$ is compact, it is not necessarily true that $T: X \to R(T)$ is compact. Here is the difficulty: $\{x^n\}$ may be a bounded sequence in X such that every convergent subsequence $\{Tx^{n_i}\} \to y \notin R(T)$. For example, consider

$T: c_0 \to l_2$ defined by $T(c_1, c_2, \ldots) = (c_1, c_2/2, c_3/3, \ldots)$

and $\{x^n\} = (\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots)$. Then $\|x^n\| = 1$ $\forall n$ so the seq. is bdd,

$Tx^n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots)$ which converges to $(1, \frac{1}{2}, \frac{1}{3}, \ldots) \notin R(T)$

(therefore every subsequence Tx^{n_i} also converges to $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$.)

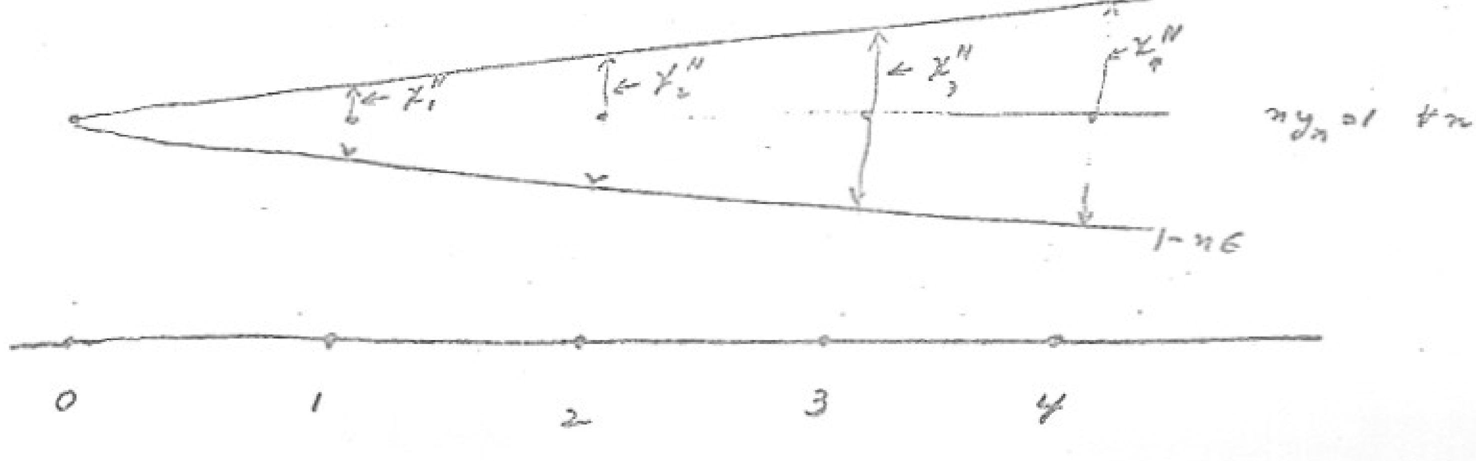
However, it can be shown for the $\tilde{T}$ of the paper, $\tilde{T}: l_2 \to l_2$ that $\tilde{T}: l_2 \to R(\tilde{T})$ remains compact. The proof is analogous to the proof for $T: l_2 \to l_2$ defined by $T(x_1, x_2, \ldots) = (x_1, x_2/2, \ldots)$ and the latter proof is given below.

Define $T: \ell_2 \to \ell_2$ by: $T(x_1, \dots, x_n, \dots) = (x_1/1, \dots, x_n/n, \dots)$

Then $y \notin R(T) \iff \sum |n y_n|^2 = \infty$. Suppose, for such a y, there is a sequence $\{x^N\}$ such that $\{Tx^N\} \to y$. We will show $\{Tx^N\}$ is not bounded. From this it follows that if $\{w^N\}$ is a bounded sequence $\ni \{Tw^N\} \to y$, $y \in R(T)$. Thus, since $T: \ell_2 \to \ell_2^{\ldots}$ is compact, so is $T: \ell_2 \to R(T)$.

Given $\epsilon > 0$ choose $N(\epsilon) \ni \sqrt{\sum_{n=1}^{\infty} \left| \frac{x_n^{N(\epsilon)}}{n} - y_n \right|^2} < \epsilon$, i.e.

$\sum_{n=1}^{\infty} \frac{|x_n^{N(\epsilon)} - n y_n|^2}{n^2} < \epsilon^2$. This implies $|x_n^{N(\epsilon)} - n y_n| < n\epsilon$ for all n and in particular $|x_n^N| > n \sup(|y_n| - \epsilon, 0)$. For the particular case $y = (1, \frac{1}{2}, \frac{1}{3}, \dots)$, the x_n^N are "close" to the ny_n in the sense of being in the "cone" indicated below. Thus $\sum |n y_n|^2 = \infty$ should lead to the construction of arbitrarily large x^N.



We make this precise as follows:

Given $A > 0$, choose $M(A) \ni \sum_{n=1}^{M(A)} |n y_n|^2 \geq 4A$.

choose $\epsilon > 0 \ni \epsilon \leq \frac{1}{2} \min_{\substack{1 \leq n \leq M(A) \\ \text{and } y_n \neq 0}} (|y_n|)$.

Then for all $y_n \ni 1 \leq n \leq M(A)$, $\sup(|y_n| - \epsilon, 0) \geq \frac{1}{2}|y_n|$

Therefore if $N(\epsilon)$ is chosen $\ni k \geq N(\epsilon) \Rightarrow \|Tx^k - y\| < \epsilon$,

$\forall k \geq N(\epsilon)$ and $\forall n \ni 1 \leq n \leq M(A)$,

$|x_n^k| \geq n \sup(|y_n| - \epsilon, 0) \geq \frac{1}{2} n |y_n|$ so

$\|x^k\| \geq \sum_{n=1}^{M(A)} \left(\frac{1}{2} n |y_n|\right)^2 = A \quad \forall k \geq N(\epsilon)$.

Thus, since A was arbitrary, $\{x^k\}$ is not bounded (and, in fact, has no bounded subsequence).

Question: Under what circumstances does a compact operator $T: X \to Y$ remain compact when Y is replaced by $R(T)$?