Journal für die reine und angewandte Mathematik
Herausgegeben von **Helmut Hasse** und **Hans Rohrbach**

Verlag Walter de Gruyter & Co., Berlin 30

Sonderabdruck aus Band 217, 1965. Seite 69 bis 78

The Range as Range Space for Compact Operators. II*)

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If X and Y are normed linear spaces and $T: X \to Y$ is a compact operator, when does $T_0: X \to R(T)$, where $R(T)$ is the range of T and T_0 is defined by $T_0 x = Tx$, remain (or fail to remain) compact? We extend the answers given in [5] to this question.

We assume throughout that the various bases are normalized, i. e. $|u_i| = 1$, all i, for the basis $\{u_i\}$. This simplifies notation and does not affect the generality of our arguments. Limits of summation from 1 to ∞ are sometimes omitted. All others have been made explicit. Operators are assumed to be bounded, linear and with domain the entire space. Refer to [4] for unspecified notational conventions.

Definition 1. Let X and Y be normed spaces and $T: X \to Y$ be compact. If the map $T_0: X \to T(X)$ defined by $T_0 x = Tx$ is compact, then T is *perfectly compact*. Otherwise T is imperfectly (or, not perfectly) compact. We call T_0 the reduced operator for T.

We use the following elementary properties of perfectly compact operators.

Lemma 2. (i) *Let W, X, Y and Z be normed spaces. If $S: W \to X$ is onto, $T: X \to Y$ is perfectly compact, and $U: Y \to Z$, then TS, UT and hence UTS, are perfectly compact.*

(ii) *Operators with finite dimensional range are perfectly compact.*

(iii) *Scalar multiples of perfectly compact operators are perfectly compact.*

(iv) *The sum of perfectly compact operators need not be perfectly compact.*

(v) *If B is bounded and P is perfectly compact, then BP is perfectly compact but PB need not be perfectly compact.*

Proof. To see the first part of (i), note that $R(T) = R(TS)$. To establish the second part, let $\{x_n\} \subset X$ be a bounded sequence. Since $T: X \to Y$ is perfectly compact, Tx_n has a subsequence $Tx_{n_i} \to Tx$ for some x in X. Since U is continuous, $UTx_{n_i} \to UTx$ also. Thus UT is perfectly compact.

Statement (ii) follows from the completeness of finite dimensional normed linear spaces. Statement (iii) follows from (i), (iv) follows from Lemma 14 below, and (v) follows from Lemma 15 (i) below.

It was shown in [5] that there are imperfectly compact operators. We next show (Theorems 5, 7, 9) that in some cases the existence (or non-existence) of imperfectly compact operators is determined by the nature of the domain space.

*) This work was supported in part by the National Science Foundation under research grant NSF-G 25058.

Definition 3. A B-space X is imperfect if for any infinite dimensional B-space Y, there always is an imperfectly compact operator $T: X \to Y$. A B-space X is perfect if for every B-space Y, all compact operators from X to Y are perfectly compact. A B-space is mixed if it is neither perfect nor imperfect.

Lemma 4. (i) *Spaces isomorphic to perfect, mixed or imperfect spaces are, respectively, perfect, mixed or imperfect.*

(ii) *Let $X = X_1 \oplus X_2$. If X_1 is imperfect, then X is imperfect. If X is perfect, then X_1 and X_2 are perfect. Conversely, if X_1 and X_2 are perfect, then X is perfect.*

(iii) *A B-space which is the continuous linear image of a perfect B-space is perfect.*

Proof. The first and third statements follows from Lemma 2(i). If $T_1: X_1 \to Z$ is imperfectly compact, then letting P denote a projection of X onto X_1, $T_1 P: X \to Z$ is imperfectly compact, which establishes the first two statements in (ii).

Now suppose that $X = X_1 \oplus X_2$ with X_1, X_2 perfect. Let $T: X \to Y$ be compact. Then $T|_{X_1}, T|_{X_2}$ are compact, hence both $(T|_{X_1})_0: X_1 \to TX_1$ and $(T|_{X_2})_0: X_2 \to TX_2$ are compact. Therefore $\overline{T(S_{X_1})}^{TX_1}$ and $\overline{T(S_{X_2})}^{TX_2}$ are compact, hence complete, so $\overline{T(S_{X_1})}^{TX}$ and $\overline{T(S_{X_2})}^{TX}$ are compact. Thus, $(T|_{X_1})_1: X_1 \to TX$ and $(T|_{X_2})_1: X_2 \to TX$ are compact. It follows that $T_0: X \to TX$ is compact, for if $P: X \to X_1$, then

$$T_0 = T_0 P + T_0(I - P) = (T_0|_{X_1}) P + (T_0|_{X_2})(I - P) = (T|_{X_1})_1 P + (T|_{X_2})_1 (I - P),$$

which is compact, for $(T|_{X_1})_1 P$ and $(T|_{X_2})_1 (I - P)$ are compact since (e. g.)

$$\overline{(T|_{X_1})_1 P(S_X)}^{TX} \subset \overline{(T|_{X_1})_1 \, |P| \, S_{X_1}}^{TX} = |P| \, \overline{(T|_{X_1})_1 (S_{X_1})}^{TX},$$

which is compact. Thus X is perfect.

Remark. We do not presently know whether there are any mixed B-spaces. Since the only perfect spaces we know of are the reflexive spaces (Theorem 9), we do not know about the converse to part of 4(ii), namely, whether if X_1 and X_2 are perfect, it implies that $X_1 \oplus X_2$ is perfect.

Theorem 5. *If X is isomorphic to any infinite dimensional complemented subspace of any abstract L-space* [3, pp. 100ff.] *(in particular if X is an abstract L-space), then X is imperfect.*

Proof. It follows from [9, page 221, Cor. 4] that $X = l_1 \oplus X_2$, hence by Lemma 4(ii), it suffices to prove the Theorem for $x = l_1$.

Every infinite dimensional B-space Z contains a closed infinite dimensional basis space Y [2]. Hence it suffices to define T from l_1 into Y as in Lemma 6 below.

Lemma 6. *Let Y be a B-space with basis $\{u_i\}$. Let $\{e_i\}$ be the standard basis for l_1. Let $T: l_1 \to Y$ be defined by $Te_i = u_1 + \frac{u_{i+1}}{2^i}$. Then*

$$Tx = (\Sigma x_i) u_1 + \frac{x_1}{2} u_2 + \frac{x_2}{2^2} u_3 + \cdots + \frac{x_n}{2^n} u_{n+1} + \cdots.$$

Let T_0 be the operator from l_1 onto $T(l_1)$ defined by $T_0 x = Tx$. Then

(a) *T is defined on all of l_1 and T is continuous.*

(b) *T, and hence T_0, is one to one.*

(c) *T is compact but T_0 is not compact.*

Proof. The first assertion follows from

$$| Tx | \leqq | \Sigma x_i | + \frac{|x_1|}{2} + \cdots + \frac{|x_n|}{2^n} + \cdots \leqq 2 | x |.$$

To see that T is compact, define the finite dimensional range operators $T^{(n)}$ by $T^{(n)}x = (\Sigma x_i) u_1 + \cdots + \frac{x_n}{2^n} u_{n+1}$. Then

$$| Tx - T^{(n)}x | = \left| \frac{x_{n+1}}{2^{n+1}} u_{n+2} + \cdots \right| \leqq \frac{|x_{n+1}|}{2^{n+1}} + \cdots \leqq \frac{|x|}{2^{n+1}},$$

so T is the uniform limit of finite dimensional range operators, hence compact.

To see that T_0 is not compact, consider the bounded sequence $\{e_i\}$. The sequence $\{T_0 e_i\}$ converges to u_1, hence so does any subsequence. But u_1 is not the image of any element in l_1, for if it were, we would have from the uniquess of the basis expansion that $\Sigma x_i = 1$, yet $\frac{x_1}{2} = 0, \ldots, \frac{x_n}{2^n} = 0, \ldots$, hence $x_1 = \cdots = x_n = \cdots = 0$, a contradiction.

Several abstract L-spaces are listed in [4, page 511]. The next theorem strongly generalizes the counter-example of [5, Example] and exhibits other imperfect spaces.

Theorem 7. *Let the B-space W be the direct sum of the B-spaces X and Y. Suppose X has a basis $\{e_i\}$ such that for some scalar sequence $\{a_i\}$, $\sup_n \left| \sum_{i=1}^{n} a_i e_i \right| < \infty$ yet $\Sigma a_i e_i$ does not define an element of X (i. e. the basis $\{e_i\}$ is not boundedly complete). Then W is imperfect. In particular, if W is a basis space with a boundedly incomplete basis $\{e_i\}$ and Z is a basis space with basis $\{u_i\}$, then there is an imperfectly compact operator $T: W \to Z$ which is diagonal relative to the pair of bases $\{e_i\}$ and $\{u_i\}$.*

Proof. It suffices by Lemma 4(ii) to prove the theorem for $W = X$. It further suffices, by the first theorem in [2], to give a proof under the additional supposition that Z has a basis $\{u_i\}$.

If $x = \Sigma c_i e_i$, we attempt to define T by $T(\Sigma c_i e_i) = \Sigma \frac{c_i}{2^i} u_i$. Now

$$\left| \sum_{i=1}^{n} \frac{c_i}{2^i} u_i \right| \leq \sum_{i=1}^{n} \frac{|c_i|}{2^i}$$

and since the sequence $\{U_m\}$ of operators on X defined by $U_m x = \sum_{i=1}^{m} c_i x_i$ is uniformly bounded [3, page 67, Theorem 1 (i)] by some constant K, $| \Sigma c_i e_i | \leq 1$ implies $| c_i | = | c_i e_i | \leq 2K$ for all i. Thus the series for Tx is absolutely convergent for all x so Tx is defined. It is evident that T is the uniform limit of the continuous finite dimensional range operators T_n defined by $T_n x = \sum_{i=1}^{n} \frac{c_i}{2^i} u_i$, hence T is compact.

To see that the reduced operator is not compact, let $\{a_i\}$ be a scalar sequence as in the hypothesis. (Normalization of the bases doesn't affect this. If $\{d_i\}$ is a not necessarily normalized basis and $\{b_i\}$ is a sequence of scalars such that $\sup_n \left| \sum_{i=1}^{n} b_i d_i \right| < \infty$ yet $\sum_{i=1}^{\infty} b_i d_i$ does not define an element of X, then $a_i = b_i | d_i |$ defines such a scalar sequence for the normalized basis $e_i = \frac{d_i}{| d_i |}$.)

Consider the bounded sequence $x^{(n)} = \sum_{i=1}^{n} a_i e_i$. If

$$\sup_n |x^{(n)}| \leq M, \quad \sup_i |a_i| = \sup_i |a_i e_i| \leq 2M \quad \text{so} \quad z = \sum_{i=1}^{\infty} \frac{a_i}{2^i} u_i$$

converges absolutely and therefore defines an element of Z. Further, $Tx^{(n)}$ converges to z. Yet z is not in $R(T)$, for, from the uniqueness of the basis expansion and the fact that T is $1-1$, the only preimage of z is $\Sigma a_i e_i$, which does not define an element of X. This proves the theorem.

Corollary 8. *The spaces $C[0, 1]$, $c_0(S)$ and $c(S)$, where S is an infinite point set, are imperfect.*

Proof. It is easy to show that the basis for $C[0, 1]$ given in [8, page 237] is not boundedly complete. In particular, there is a scalar sequence $\{a_i\}$ such that $\sum_{i=1}^{n} a_i e_i$ assumes the values $f_1, f_2, \ldots, f_k, \ldots$ given by $f_k(t) = 2^k t$, $0 \leq t \leq 2^{-k}$, $f_k(t) = 1$, $2^{-k} < t \leq 1$. Then f_k converges pointwise to the characteristic function of $[0, 1]$. Alternately, one can reduce the proof for $C[0, 1]$ to that for c_0 by noting that $C[0, 1]$ contains a complemented copy of c_0 [9, Theorem 4].

To see that the usual bases for c_0 and for c are not boundedly complete, consider the sequence $\{x^{(n)}\}$ of elements of c_0 defined by $x_i^{(n)} = (-1)^i$ if $i \leq n$, $x_i^{(n)} = 0$ if $i > n$, which corresponds to the scalar sequence of a_i defined by $a_i = (-1)^i$, all i. Whenever S is an infinite point set, $c_0(S)$ and $c(S)$ contain complemented copies of c_0 and c respectively. Hence the theorem holds for $c_0(S)$ and $c(S)$ as well.

Remark. If there were an infinite dimensional B-space Z such that every compact operator from l_∞ to Z were perfectly compact, then it would follow from Lemma 4(ii) and Corollary 8, letting $X = l_\infty$ and $X_1 = c_0$, that there can be no continuous projection from l_∞ onto c_0, a well-known result of Phillips [10].

This illustrates a possible application of the theory. Whenever we can find a (non-reflexive) perfect or mixed space X with an imperfect subspace Z, we then know that Z is not complemented in X.

Theorem 9[1]**.** *If X is reflexive, Y is a normed linear space, and $T: X \to Y$ is compact, then $T_0: X \to R(T)$ is compact. In particular, reflexive spaces are perfect.*

Proof. Replace Y by its completion, Y_1. The operator T remains compact. By Ringrose's Theorem [11, Theorem 3.5], T is continuous when restricted to $(S_X, \tau_{X^*}) \to Y_1$ where S_X is the closed unit sphere in X and τ_{X^*} is the X^* topology. Since X is reflexive, (S_X, τ_{X^*}) is compact so $T(S_X)$ is compact in Y_1. Therefore it is compact as a subset of $R(T)$. Therefore $T(S_X) = \overline{T(S_X)}^{R(T)}$ so the latter is compact. Thus T_0 is compact and the theorem is established.

We do not presently know whether there are non-reflexive perfect spaces.

Theorems 7 and 9 together show that if X is a reflexive basis space, every basis for X is boundedly complete. This is part of a well-known theorem of James [6].

Theorem 9 shows us that when $1 < p < \infty$, the Theorem and Corollary in [5] are true for all compact operators, rather than just the compact diagonal operators.

[1]) This theorem is due to R. Whitley. The proof above is due to E. Lacey. I wish to thank E. Lacey and R. Whitley for helpful comments and observations.

In [5, Theorem] we call an operator T between sequence spaces X and Y diagonal if there is a scalar sequence $\{c_i\}$ such that T is defined by $Tx = y$ where $x = \{x_i\}$ is an arbitrary element of X and $y = \{c_i x_i\}$. We did not explicitly characterize the compact diagonal operators in terms of the associated scalar sequence $\{c_i\}$. It is not difficult to do this for the spaces occuring in [5, Theorem], namely, $X = l_p$, $Y = l_q$, $1 \leqq p, q \leqq \infty$.

The next lemma includes the characterization of compact diagonal operators in the two cases which still retain their interest after the discovery of Theorem 9, namely, $p = 1$ and $p = \infty$. Part (c) of the Lemma is included for completeness. It is not needed specifically for the two cases of interest for our problem.

Lemma 10. *If $X = l_p$ and $Y = l_q$, a diagonal operator T associated with the sequence $\{c_i\}$ is compact if and only if*

(a) *$1 \leqq p \leqq q \leqq \infty$ and $\{c_i\}$ is an element of c_0.*

(b) *$p = \infty > q \geqq 1$ and $\{c_i\}$ is an element of l_q.*

(c) *$\infty > p > q \geqq 1$ and $\{c_i\}$ is an element of $l_{p'}$, where p' is defined by the equation*

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q}.$$

Proof. For part (a), suppose first that $q < \infty$. It is easily seen that a necessary condition that a diagonal operator be bounded is that $\{c_i\} \in l_\infty$. In this case, from Jensen's inequality we have

$$(\Sigma \mid c_i x_i \mid^q)^{\frac{1}{q}} \leqq \sup_i \mid c_i \mid (\Sigma \mid x_i \mid^q)^{\frac{1}{q}} \leqq \sup_i \mid c_i \mid (\Sigma \mid x_i \mid^p)^{\frac{1}{p}}.$$

Hence $\mid T \mid \leqq \sup_i \mid c_i \mid$.

Choosing $x = e_i$ shows that $\mid T \mid \geqq \sup_i \mid c_i \mid$, where e_i is the i-th basis element of l_p. Hence the bounded diagonal operators are precisely those with $\{c_i\} \in l_\infty$. Further, $\mid T \mid = \sup_i \mid c_i \mid$.

If $\{c_i\} \in c_0$, the operator is the uniform limit of finite dimensional range operators and therefore is compact. If $\{c_i\} \notin c_0$, there is an $\varepsilon > 0$ and a subsequence $\{c_{n_i}\}$ such that $\mid c_{n_i} \mid \geqq \varepsilon$. Then $\{e_{n_i}\}$ is a bounded sequence whose image has no convergent subsequence and the operator is not compact. Therefore the compact diagonal operators are precisely those such that $\{c_i\}$ is in c_0.

This proof also yields the case $1 \leqq p < q = \infty$ after an obvious modification of the inequality chain.

Next we establish the remaining case of (a), $p = q = \infty$, and (b). The stated conditions on $\{c_i\}$ imply that T is the uniform limit of finite dimensional range operators, and therefore compact. Conversely, it can be seen easily that the operator is not compact if the conditions are not fulfilled.

The fact that the diagonal operators satisfying the condition in part (c) of the lemma are bounded follows from [4, VI. 11. 1]. They are also compact because $\infty > p > q \geqq 1$ implies all bounded operators are compact [1, page 700].

Conversely, suppose that T is a diagonal operator such that $\{c_i\}$ is not in $l_{p'}$, i. e., $\Sigma \mid c_i \mid^{p'} = \infty$. Let $\sum_{i=1}^{N} \mid c_i \mid^{p'} = K_N$ and let $x_i = (\operatorname{sgn} c_i) \mid c_i \mid^{\left(\frac{p'}{q} - 1\right)}$ where $\operatorname{sgn} c_i$ is defined as 0 when $c_i = 0$ and as $e^{-i\theta}$ when $c_i \neq 0$ and $c_i = re^{i\theta}$.

Then $\sum_{i=1}^{n} |x_i|^p = \sum_{i=1}^{n} |c_i|^{\left(\frac{p'}{q}-1\right)p} = \sum_{i=1}^{n} |c_i|^{p'} = K_n$, so each member of the sequence of vectors $\{x^{(n)}\}$ defined by

$$x^{(n)} = \frac{\sum_{i=1}^{n} x_i e_i}{K_n^{\frac{1}{p}}}$$

has norm 1.

But

$$| Tx^{(n)} |^q = \frac{\sum_{i=1}^{n} | c_i x_i |^q}{K_n^{\frac{1}{p}}} = \frac{\sum_{i=1}^{n} | c_i |^{p'}}{K_n^{\frac{1}{p}}} = (K_n)^{1-\frac{1}{p}}$$

so $\sup_n | Tx^{(n)} | = \infty$ and T is unbounded.

The proof of Theorem 9 suggests that we study the stronger (Lemma 12(ii)) notion of full compactness (Definition 11) as a possible approach to investigating perfect compactness.

Definition 11. Let X and Y be normed linear spaces. An operator $T : X \to Y$ is *fully compact* if $T(S_X)$ is compact.

Lemma 12. (i) *Let W, X, Y and Z be normed linear spaces. If $S : W \to X$ maps the unit sphere onto a closed set, $T : X \to Y$ is fully compact, and $U : Y \to Z$, then TS, UT and hence UTS are fully compact.*

(ii) *Fully compact operators are perfectly compact.*

(iii) *A compact operator with finite dimensional range need not be fully compact. Hence, perfect compactness does not imply full compactness.*

(iv) *Scalar multiples of fully compact operators are fully compact.*

(v) *The sum of two fully compact operators need not be fully compact.*

(vi) *If F is fully compact and B is bounded, FB need not be fully compact.*

(vii) *The uniform limit of fully compact operators need not be fully compact.*

Proof. The first statement is evident and the second is established in the course of the proof of Theorem 9.

We establish (iii) by taking $X = c_0$, $Y \neq (0)$ and letting e be any non-zero vector in Y. The operator $T : X \to Y$ defined by $T(x_i) = \Sigma \frac{x_i}{2^i} e$ has norm one and has one dimensional range. It is not fully compact because the set $T(S_X) = \{ce : |c| < 1\}$, where c is a scalar, and that set is not compact.

Statement (iv) follows from (i).

Statement (v) follows from (ii) and Lemma 14 below.

Statement (vi) follows from (ii) and Lemma 15(i) below. The last statement follows from (ii) and Lemma 15(iii) below.

Remark. Analogous to Definition 3 for perfect spaces, we could define a B-space X to be full if for each B-space Z, every compact operator $T : X \to Z$ is fully compact. The concept of a full space has no interest for the following reasons. The proof of Theorem 9 shows that reflexive spaces are full. Conversely, it follows from James [7)] that a full space is reflexive.

The next theorem provides useful examples of fully compact operators.

Theorem 13. *Let X and Y be basis spaces and suppose X has a boundedly complete basis $\{u_i\}$. Then every compact diagonal operator is fully compact.*

Proof. Without loss of generality we can assume that $|u_i| = 1$, all i. We can further assume that X and Y have the form of the canonically associated isomorphic spaces described in [3, page 67, Theorem 1]. This does not alter our assumptions that $|u_i| = 1$, all i, and that $\{u_i\}$ is boundedly complete.

Suppose first that all c_i are non-zero. If $x^{(n)}$ is any sequence in S_X, it suffices to show that $Tx^{(n)} \to y = \{y_i\}$ implies that y is in $T(S_X)$, i. e., that $\left\{\frac{y_i}{c^i}\right\}$ is in S_X. Since $Tx^{(n)} \to y$ and the coordinate functionals are continuous, it follows that $\lim_n c_i x_i^{(n)} = y_i$ for each i. Hence for each positive integer M,

$$\left|\sum_{i=1}^{M} \frac{y_i}{c_i} u_i\right| \leq \left|\sum_{i=1}^{M}\left(x_i^{(n)} - \frac{y_i}{c_i}\right) u_i\right| + \left|\sum_{i=1}^{M} x_i^{(n)} u_i\right| \leq \left|\sum_{i=1}^{M}\left(x_i^{(n)} - \frac{y_i}{c_i}\right) u_i\right| + 1 \to 1 \text{ as } n \to \infty.$$

The last inequality follows from the monotonicity of the basis $\{u_i\}$. The last limit again follows from the continuity of the coordinate functionals.

Since X is boundedly complete, we may conclude that $\left\{\frac{y_i}{c_i}\right\}$ defines an element of X. The inequality shows that $\left\{\frac{y_i}{c_i}\right\}$ is in S_X.

If some of the $c_i = 0$, say $\{c_{i_j}\}$, then if $Tx^{(n)} \to y$, it follows that all y_{i_j} are zero. The proof then proceeds as above, where all sums avoid the indices $\{i_j\}$.

Remark. The Theorem and proof are valid for $X = l_\infty$, where each u_i is replaced by the characteristic function of the set $\{i\}$.

Lemma 14. *The sum of two fully compact operators need not be perfectly compact.*

Proof. In Lemma 6, let $Y = l_1$ and consider the operator T defined there. We can write $T = T_1 + T_2$ where $T_1 = (\Sigma x_i) u_1$ and $T_2 = T - T_1$. To see that T_1 is fully compact, note that $|T_1 x| = |\Sigma x_i| \leq \Sigma |x_i| = |x|$, so $T_1(S_X)$ is contained in the compact set $K = \{c u_1 : |c| \leq 1\}$, where c is a scalar. However, $T_1(c u_1) = c u_1$ so $T_1(S_X) = K$.

Now $T_2 = I U_2$ where U_2 is the diagonal operator defined by

$$U_2 x = \frac{x_1}{2} u_1 + \cdots + \frac{x_n}{2^n} u_n + \cdots$$

and I is the isometry defined by $I u_i = u_{i+1}$. By Theorem 13, U_2 is fully compact. Hence by Lemma 12(i), $I U_2 = T_2$ is fully compact. Since T is not perfectly compact, the Lemma is proven.

The next Lemma supplies additional evidence of the pathology associated with the concepts of full compactness and perfect compactness.

Lemma 15. *Let X, Y and Z be normed linear spaces.*

(i) *If $B: X \to Y$ is bounded and $F: Y \to Z$ is fully compact, then FB need not be perfectly compact.*

(ii) *If T^* is fully compact T may fail to be perfectly compact.*

(iii) *The uniform limit of fully compact operators need not be perfectly compact.*

Proof. To establish (i), let B be the injection i of c_0 into l_∞ defined by $i(x) = x$. Let $F: l_\infty \to l_1$ be defined by $F(\{x_i\}) = \frac{x_i}{2^i} e_i$. By the remark following the proof of Theorem 11, F is fully compact. But FB is the operator T constructed in the proof of Theorem 7 and is not perfectly compact.

For an example of an operator T which is not perfectly compact, but whose conjugate T^* is fully compact, let $T: c_0 \to c_0$ be defined by $T(\{x_i\}) = \left\{\frac{x_i}{2^i}\right\}$. Then $T^*: l_1 \to l_1$ is given by $T^*(\{x_i\}) = \left\{\frac{x_i}{2^i}\right\}$ and is fully compact by Theorem 13, which establishes (ii).

It is easy to see that each of the operators $T^{(n)}$ defined by $T^{(n)}(\{x_i\}) = \sum_{i=1}^{n} \frac{x_i}{2^i} e_i$ is fully compact and since $T^{(n)} \to T$ uniformly, (iii) is proven.

Definition 16. If X and Y are normed linear spaces, an operator $T: X \to Y$ is totally bounded if $T(S_X)$ is a totally bounded set.

The next lemma lists some elementary facts about the relation between totally bounded operators and compact operators. The proofs are straightforward and are omitted.

Lemma 17. (i) *If Y is complete, T is totally bounded if and only if it is compact.*

(ii) *Compact operators are totally bounded.*

(iii) *If X and Y are B-spaces, then T is compact if and only if the reduced operator T_0 (see Definition 1) is totally bounded.*

(iv) *If X and Y are B-spaces, T is imperfectly compact if and only if the associated is totally bounded but not compact.*

(v) *The totally bounded operators $T(X, Y)$ are a subspace of the bounded operators $B(X, Y)$. When $X = Y$, $T(X, X)$ is a two-sided ideal in $B(X, X)$. The subspace $T(X, Y)$ is closed in $B(X, Y)$ with the uniform topology. The compact operators $K(X, Y)$ are dense in $T(X, Y)$. If Y is complete, $T(X, Y) = K(X, Y)$. If Y is not complete, $K(X, Y)$ may operator T_0 or may not be properly contained in $T(X, Y)$.*

The concept of totally bounded operator seems to be the natural generalization to normed linear spaces, from B-spaces, of the concept of compact operator. Lemma 17 shows us that the study of imperfect compactness is equivalent to the study of the difference between totally bounded and compact operators.

Theorem 18[2]**.** *If X and Y are normed linear spaces and $T: X \to Y$ is totally bounded, T^* is fully compact.*

Proof. If we complete X and Y and extend T to the completion, the extended operator is compact and its conjugate is also T^*. Thus we may assume that X and Y are complete and that T is compact.

By [4, VI. 5. 6], T^* is continuous when restricted to $(S_{X^*}, \tau_X) \to X^*$, where τ_X is the relativized X topology. By Alaoglu's theorem [4, V. 4. 2], (S_{X^*}, τ_X) is compact. The proof continues in the same way as that of Theorem 9.

One might wonder whether Theorem 18 will provide examples where X is not reflexive and all the compact operators from X to a fixed Y are fully compact. The result of [7] is that when X is a B-space which is not reflexive, there are continuous linear functionals $x^* \in X^*$, which do not attain their supremum on S_X. Thus we can always manufacture one dimensional operators which are not fully compact.

[2]) This Theorem is due to E. Lacey.

Alternately, Theorem 18 will provide such examples only if X and Y are conjugate spaces and further, all the compact operator from X to Y are conjugate operators. The next theorem shows us, however, that all the compact operators are conjugates if X is reflexive, hence no new examples are obtained from this line of argument.

Let X and Y be Banach spaces. Let $B(X, Y)$ be the set of bounded operators from X to Y and $K(X, Y)$ the set of compact operators. Let $\Phi : B(X, Y) \to B(Y^*, X^*)$ be defined by $\Phi(T) = T^*$. It is well known that Φ is a linear isometry into, and that $\Phi(K(X, Y)) \subset K(Y^*, X^*)$.

Theorem 19. *If X is a normed linear space and Y is a Banach space, the following statements are equivalent:*

(i) *Y is reflexive.*

(ii) $\Phi(B(X, Y)) = B(Y^*, X^*)$.

(iii) $\Phi(K(X, Y)) = K(Y^*, X^*)$.

Proof. First we show that if Y is not reflexive, $\Phi(B(X, Y)) \not\supset K(Y^*, X^*)$. Let $Tx = x_0^*(x)\, y_0$ define a one-dimensional range operator in $B(X, Y)$. Then

$$T^* y^* x = y^* Tx = x_0^*(x)\, y^*(y_0) \text{ so } T^* y^* = y^*(y_0)\, x_0^* = (J_Y y_0)\,(y^*)\, x_0^*$$

and all one-dimensional range operators in $B(Y^*, X^*)$ of the form $(J_Y y_0)\,(\cdot)\, x_0^*$ are adjoints.

Suppose $y_0^{**} \notin J_Y(y)$. Consider U^* defined by $U^* y^* = y_0^{**}(y^*)\, x_0^*$. Then U^{**} is defined by $U^{**} x^{**} = x^{**} U^* = x^{**}(x_0^*)\, y_0^{**}$. If U^* is the adjoint of an operator U, then since $U^{**} J_X = J_Y U$ [4, page 479, Lemma 6], if

$$x^{**} = J_X x,\ (J_X x)\,(x_0^*)\, y_0^{**} = x_0^*(x)\, y_0^{**} \in J_Y(Y).$$

Second, we show that if Y is reflexive, Φ is onto. Let $U^* \in B(Y^*, X^*)$. Consider $U^{**} \in B(X^{**}, Y^{**})$ defined by $U^{**} x^{**} = x^{**} U^*$. Then the equation $U^{**} J_X = J_Y U$ determines an operator $T = J_Y^{-1} U^{**} J_X \in B(X, Y)$ because $R(U^{**}) \subset Y^{**} = J_Y(Y)$.

We compute T^* and hope $T^* = U^*$. $T^* y^* = y^*(J_Y^{-1} U^{**} J_X)$ so

$$T^* y^* x = y^*(J_Y^{-1} U^{**} J_X x) = U^{**}(J_X x)\, y^* = (J_X x)\, U^* y^*$$

so $U^* y^* = T^* y^*$ and $U^* = T^*$.

Third, note that equality in (ii) implies equality in (iii) because T is compact if T^* is compact.

The equivalence follows from the three statements we have proved.

Remark. When Theorem 19 is used in conjunction with operator representation theorems which appear in the literature, it yields a number of operator representation theorems which do not seem to appear in the literature. For example, it is well-known [12, page 278, Example 3] that if $1 < q < \infty$, there is a (canonical) $1 - 1$ correspondence between the elements A of $K(l_1, l_q)$ and the infinite matrices (a_{ij}) satisfying

$$\| A \| = \sup_j \left(\sum_{i=1}^{\infty} | a_{ij} |^q \right)^{\frac{1}{q}} < \infty \text{ and also } \lim_n \sum_{i=n}^{\infty} | a_{ij} |^q = 0$$

uniformly in j.

A corresponding representation theorem for $K(l_q, l_\infty)$, $1 < q < \infty$, does not seem to be well-known [4, page 548, Table VI. B]. We prove such a theorem to illustrate the idea.

Theorem 20. *If $1 < q < \infty$, there is a (canonical) 1—1 correspondence between the elements A of $K(l_q, l_\infty)$ and the infinite matrices (a_{ij}) satisfying*

$$\| A \| = \sup_i \left(\sum_{j=1}^{\infty} | a_{ij} |^q \right)^{\frac{1}{q}} < \infty \text{ and also } \lim_n \sum_{j=n}^{\infty} | a_{ij} |^q = 0$$

uniformly in i.

Proof. By [12, page 221, Ex. 6], the conjugate of operators in $B(l_1, l_{q'})$ are represented by transpose matrices. By Theorem 19, Φ maps $K(l_1, l_{q'})$ onto $K(l_q, l_\infty)$ so $K(l_q, l_\infty)$ is represented precisely by these transpose matrices. Applying this to the theorem for $K(l_1, l_q)$ gives Theorem 20.

Using Theorem 19, the only fact we require in addition to a known representation theorem for $B(X, Y)$, $K(X, Y)$, $B(Y^*, X^*)$ or $K(Y^*, X^*)$, with Y reflexive, is that we can deduce, from the given operator or conjugate operator, the representation of the corresponding conjugate or preconjugate operator. If this holds, representation theorems which are unlisted in [4, Tables VI. A. and VI. B.] can be deduced from those that are listed, as follows:

$B(L_p, L_\infty)$ from $B(L_1, L_p)$, $1 < p < \infty$, provided $L_1^* = L_\infty$.

$B(l_p, L_\infty)$ from $B(L_1, l_p)$, $1 < p < \infty$, provided $L_1^* = L_\infty$.

$B(l_p, L_p)$ from $B(L_p, l_p)$, $1 < p < \infty$.

$B(c, L_p)$ and $B(c_0, L_p)$ from $B(L_p, l_1)$, $1 < p < \infty$.

$K(l_p, L_\infty)$ from $K(L_1, l_p)$, $1 < p < \infty$, provided $L_1^* = L_\infty$.

$K(c, L_p)$ and $K(c_0, L_p)$ from $B(L_p, l_1)$, $1 < p < \infty$.

D. R. Arterburn has developed a similar theory of weakly perfectly compact operators, to appear. He kindly supplied the proof of the third statement in Lemma 4 (ii). A. Carver informs us that he independently had proved a theorem (unpublished) equivalent to Theorem 19.

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Eingegangen 2. November 1963