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The Range as Range Space for Compact Operators*)

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It was asserted in [4], page 768, end of section 3, that certain compact linear operators from a Banach space to a Banach space remained compact when their original (complete) range spaces were reduced to the ranges of the operators. Examples of the existence of compact operators in certain states were a consequence of this assertion. However, unlike continuous operators, which always remain continuous under this reduction in range space, compact operators do not always remain compact under such a change, as the following example shows.

Example. Define $T: c_0 \to l_2$ by $T(x_1, x_2, \ldots, x_n, \ldots) = \left(x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots\right)$. It is easy to see that T is a limit of compact operators with finite dimensional range, whence T is compact. Let $\{x^{(n)}\}$ be the sequence of elements in c_0 defined by $x_i^{(n)} = 1$ if $i \leq n$, $x_i^{(n)} = 0$ if $i > n$. Then $\|x^{(n)}\| = 1$ for all n, so the sequence is bounded. Now $Tx^{(n)} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)$ and this converges to $z = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right)$, which is not in the range of T. Every subsequence $\{Tx^{(n_i)}\}$ also converges to z, therefore when l_2 is replaced by the range of T, the corresponding operator T_0 is not compact.

Remarks. i) This example shows that a limit of compact operators is not necessarily compact if the range space is not complete. Note that T_0 is the limit of compact operators with finite dimensional range.

ii) Since T_0' may be identified with T' in the obvious way and T' is compact, T_0' is also compact. Hence the example also shows that if the range space is not complete, a bounded operator which is not compact may still have a conjugate which is compact.

Fortunately, the operators in question in [4] do remain compact, as asserted there, when their range space is reduced to their range. This is shown in the following theorem. The original unmodified operators are listed below for convenience.

1. $X = Y = l_2$; T is defined by $Tu_k = 2^{1-k} u_k$, where $\{u_k\}$ is the usual basis. The reduced operator is in state (I_2, II_2).

2. $X = l_1$, $Y = l_2$; T is defined by $Tu_k = 2^{1-k} u_k$. The reduced operator is in state (I_2, III_2).

3. $X = Y = l_2$; T is defined by $Tu_1 = 0$, $Tu_k = 2^{1-k} u_{k-1}$, $k = 2, 3, \ldots$. The reduced operator is in state (I_2, III_2).

Theorem. *Let $T: l_p \to l_q$, $1 \leq p, q \leq \infty$, be the compact operator defined by $T\{x_k\} = \{c_k x_k\}$, where $\{x_k\} \in l_p$ and $\{c_k\}$ is a sequence of scalars such that $\{c_k x_k\}$ is in*

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l_p for all $\{x_k\} \in l_p$. If $R(T)$ is the range of T and T_0: $l_p \to R(T)$ is the reduced operator corresponding to T, then T_0 is also compact.

Proof. First suppose all c_k are non-zero. If $\{x^{(n)}\}$ is any sequence of elements in l_p such that $\| x^{(n)} \| \leq 1$ for all n, it suffices to show that $Tx^{(n)} \to y = \{y_k\}$ implies that y is in $R(T)$, i. e., that $\left\{\frac{y_k}{c_k}\right\} \in l_p$.

Case 1. $p \neq \infty$.

Since $Tx^{(n)} \to y$, it follows that $c_i x_i^{(n)} \to y_i$ as $n \to \infty$. Hence given any positive integer M,

$$(*) \qquad \left(\sum_{i=1}^{M} \left|\frac{y_i}{c_i}\right|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{M} \left|x_i^{(n)} - \frac{y_i}{c_i}\right|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{M} \left|x_i^{(n)}\right|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{i=1}^{M} \left|x_i^{(n)} - \frac{y_i}{c_i}\right|^p\right)^{\frac{1}{p}} + 1 \to 1 \quad \text{as} \quad n \to \infty.$$

Thus $\sum_{i=1}^{M} \left|\frac{y_i}{c_i}\right|^p \leq 1$ and since M was an arbitrary positive integer, we may conclude that $\left\{\frac{y_i}{c_i}\right\}$ is in l_p.

Case 2. $p = \infty$. Replace each sum in (*) by the largest term appearing in the sum.

If some of the $c_k = 0$, say $\{c_{k_j}\}$, then if $Tx^{(n)} \to y$, it follows that all y_{k_j} are zero. The proof then proceeds as above, where all sums avoid the indices $\{k_j\}$.

Corollary. *A compact symmetric operator on separable Hilbert space remains compact when the range space is reduced to the range of the operator.*

Proof. The characterization of compact symmetric operators given in [2], p. 336, shows that they satisfy the hypotheses of the theorem.

The example and theorem together suggest the following interesting, and to our knowledge unanswered, question.

Open Question. If X and Y are normed linear spaces and T: $X \to Y$ is a compact linear operator, when does T_0: $X \to R(T)$, defined by $T_0 x = Tx$, remain compact?

Four of the examples of possible states, given in [4] for the compact operator state diagram, and which include two of the three examples listed above, can be used in conjunction with examples from [3] to construct the remaining examples required for the state diagram for bounded and unbounded linear operators (cf. [3] and [1]).

Consider the following construction. Let T_1: $X_1 \to Y_1$ be in state (A_a, B_b) and let T_2: $X_2 \to Y_2$ be in state (C_c, D_d). Let $T_1 \times T_2$: $X_1 \times X_2 \to Y_1 \times Y_2$ be defined by $(T_1 \times T_2)(x_1, x_2) = (T_1 x_1, T_2 x_2)$. The norm on $X_1 \times X_2$ is any norm suitable for the product topology, for example $\|(x_1, x_2)\| = \|x_1\| + \|x_2\|$. Similarly for $Y_1 \times Y_2$. It is easily verified that the state of $T_1 \times T_2$ is precisely $(A_a, B_b) \times (C_c, D_d) = (E_e, F_f)$, where E is the greater of A and C, e is the greater of a and c, F is the greater of B and D, and f is the greater of b and d. Note that limitations on the squares of the state diagram which correspond to T_1 and T_2, such as "Y is not complete", are inherited by the $T_1 \times T_2$ example. We indicate this by writing, for example $(I_3, III_2)\,Y = (I_3, III_1) \times (I_1, II_2)\,Y$, where Y indicates that Y is incomplete since the state is impossible if Y is complete. If, however, these restrictions are all already present in the square corresponding to $T_1 \times T_2$, then it is a "best possible" example. We note that whether X or Y is inseparable is irrelevant for the bounded and the unbounded operator state diagrams.

We take as our basic stock of simple examples those given in [4] for states $(I_2, II_2)\,Y$; $(I_2, III_2)\,X_r, Y$; (II_2, II_2); $(II_2, III_2)\,X_r$; and those given in [3] for states (I_1, I_1); $(II_1, I_1)\,X$; (I_3, III_1), (III_1, I_3) and $(I_2, III_1)\,X$. Then our assertion follows from the equations:

$(III_2, II_3) = (II_2, II_2) \vee (III_1, I_3)$; $(II_3, III_2) = (II_2, II_2) \vee (I_3, III_1)$; $(III_3, III_3) = (III_2, II_3) \vee (II_3, III_2)$; $(II_2, III_1)\,X = (II_2, II_2) \vee (I_2, III_1)\,X$; $(II_3, III_1)\,X = (I_3, III_1) \vee (II_2, III_1)\,X$; $(I_3, III_2)\,Y = (I_3, III_1) \vee (I_2, II_2)\,Y$; $(III_2, III_3)\,X_r = (III_2, II_3) \vee (II_2, III_2)\,X_r$.

Remarks. In [3], the example given of an operator in state $(II_2, III_1)\,X$ does not satisfy the desired condition that Y be reflexive. Our example, constructed above, does satisfy this requirement.

This "lattice" construction we have given should prove a useful tool in manufacturing examples for the various past and future state diagrams.

References

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