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In a previous paper on statistical metric spaces [3] it was shown that a statistical metric induces a natural topology for the space on which it is defined and that with this topology a large class of statistical metric (briefly, SM) spaces are Hausdorff spaces.

In this paper we show that this result (Theorem 7.2 of [3]) can be considerably generalized. In addition, as an immediate corollary of this generalization, we prove that with the given topology a large number of SM spaces are metrizable, i.e., that in numerous instances the existence of a statistical metric implies the existence of an ordinary metric.

**THEOREM 1.** Let $(S, \mathcal{F})$ be a statistical metric space, $\mathcal{F}$ the two-parameter collection of subsets of $S \times S$ defined by

\[ \mathcal{F} = \{ U(\varepsilon, \lambda); \varepsilon > 0, \lambda > 0 \}, \]

where

\[ U(\varepsilon, \lambda) = \{(p, q); p, q \in S \text{ and } F_{p, q}(\varepsilon) > 1 - \lambda \}, \]

and $T$ a two-place function from $[0, 1] \times [0, 1]$ to $[0, 1]$ satisfying $T(a, b) \geq T(a, b)$ for $a \geq a, b \geq b$ and $\sup_{x \in S} T(x, x) = 1$. Suppose further that for all $p, q, r$ in $S$ and for all real numbers $x, y$, the Menger triangle inequality

(1) \[ F_{p, q}(x + y) \geq T(F_{p, q}(x), F_{q, r}(y)) \]

is satisfied. Then $\mathcal{F}$ is the basis for a Hausdorff uniformity on $S \times S$.

**Proof.** We verify that the $U(\varepsilon, \lambda)$ satisfy the axioms for a basis for a Hausdorff (or separated) uniformity as given in [2; p. 174-180] (or in [1; II, § 1, n° 1]).

(a) Let $\Delta = \{(p, p); p \in S\}$ and $U(\varepsilon, \lambda)$ be given. Since for any $p \in S$, $F_{p, p}(\varepsilon) = 1$, it follows that $(p, p) \in U(\varepsilon, \lambda)$. Thus $\Delta \subset U(\varepsilon, \lambda)$.

(b) Since $F_{p, q} = F_{q, p}$, $U(\varepsilon, \lambda)$ is symmetric.

(c) Let $U(\varepsilon, \lambda)$ be given. We have to show that there is a $W \in \mathcal{F}$ such that $W \circ W \subset U$. To this end, choose $\varepsilon' = \varepsilon / 2$ and $\lambda'$ so small that $T(1 - \lambda', 1 - \lambda') \geq 1 - \lambda$. Suppose now that $(p, q)$ and $(q, r)$ belong to $U(\varepsilon, \lambda)$.

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1 These considerations have led to the study of SM spaces which are not metrizable as well as to other natural topologies for SM spaces, questions which will be investigated in a subsequent paper.

2 The terminology and notation are as in [3].

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Thus $(p, r) \in U(\varepsilon, \lambda)$. But this means that $W \circ W \subset U$.

(d) The intersection of $U(\varepsilon, \lambda)$ and $U(\varepsilon', \lambda')$ contains a member of $\mathcal{W}$, namely $U(\min(\varepsilon, \varepsilon'), \min(\lambda, \lambda'))$.

Thus $\mathcal{W}$ is the basis for a uniformity on $S \times S$.

(e) If $p$ and $q$ are distinct, there exists an $\varepsilon > 0$ such that $F_p(\varepsilon) \neq 1$ and hence $\varepsilon, \lambda_0$ such that $F_p(\varepsilon) = 1 - \lambda_0$. Consequently $(p, q)$ is not in $U(\varepsilon, \lambda)$ and the uniformity generated by $\mathcal{W}$ is separated, i.e., Hausdorff.

Note that the theorem is true in particular for all Menger spaces in which $\sup_{x < 1} T(x, x) = 1$. However, it is true as well for many $SM$ spaces which are not Menger spaces.

**Corollary.** If $(S, \mathcal{F})$ is an $SM$ space satisfying the hypotheses of Theorem 1, then the sets of the form $N_p(\varepsilon, \lambda) = \{ q: F_p(\varepsilon) > 1 - \lambda \}$ are the neighborhood basis for a Hausdorff topology on $S$.

**Proof.** These sets are a neighborhood basis for the uniform topology on $S$ derived from $\mathcal{W}$.

**Theorem 2.** If an $SM$ space satisfies the hypotheses of Theorem 1, then the topology determined by the sets $N_p(\varepsilon, \lambda)$ is metrizable.

**Proof.** Let $\{(\varepsilon_n, \lambda_n)\}$ be a sequence that converges to $(0, 0)$. Then the collection $\{U(\varepsilon_n, \lambda_n)\}$ is a countable base for $\mathcal{W}$. The conclusion now follows from [2; p. 186].

Theorem 2 may be restated as follows: Under the hypotheses of Theorem 1, there exist numbers $\delta(p, q)$ which are determined by the distance distribution functions $F_p$ in such a manner that the function $\delta$ is an ordinary metric on $S$. Loosely speaking, if the statistical distances have certain properties, then certain numerical quantities associated with them have the properties of an ordinary distance. In a given particular case such quantities might be the means, medians, modes, etc. For example, most of the particular spaces studied in [3] satisfy the hypotheses of Theorem 2, hence are metrizable. Indeed, as was shown in [3], in a simple space, the means (when they exist), medians, and modes (if unique) of the statistical distances each form metric spaces; and similarly, in a normal space, the means of the $F_p$ form a (generally discrete) metric space. What Theorem 2 now tells us is that in many (though not all) $SM$ spaces we can expect results of this general nature to hold.
REFERENCES


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