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Mathematics of Gambling

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Last time we briefly discussed payoff odds for simulated horse racing on a cruise ship. This month, as promised, I'll illustrate how the payoff odds are calculated in these races.

To start, suppose \$1 is bet on the first horse, \$2 on the second horse, \$3 on the third, through \$6 on the sixth horse. The pool has $\$1 + \$2 + \ldots + \$6 = \$21$. Assume there is no track take. This means all the money is returned to the players who bet on the winning horse.

If the first horse wins, the \$1 bet gets \$21. The payoff is 21 for 1, also known as 20 to 1 ("odds for one" are always one more than "odds to one"). The payoffs for each \$1 bet are for horses 2, 3, 4, 5 and 6, respectively; twenty-one dollars for \$2, \$21 for \$3, \$21 for \$4, \$21 for \$5 and \$21 for \$6. (Throughout this article I will omit the \$ signs.) If instead there was a "track take" and the operators decided to keep \$6, the payoffs on horses 1 to 6 would be: 15 for 1; 15 for 2; 15 for 3; 15 for 4; 15 for 5 and 15 for 6, respectively. Note that I could have written 5 for 1 instead of 15 for 3, 7 for 2 instead of 21 for 6, etc. Instead, I wanted to preserve the pattern of the numbers to make them easier to follow.

A System

In the game, the track take was 20% or more. If there were no track take, we'd have the following winning system: Simply bet equal amounts on each of the six horses. You will have an advantage, a positive mathematical expectation.

Suppose you bet \$1 on each of the six horses. The amount bet on each horse is increased by \$1 and the pool is increased by \$6. The payoffs on the horses become 27 for 2; 27 for 3; 27 for 4; 27 for 5; 27 for 6; and 27 for 7, respectively. Your \$1 bet will win \$13.50, \$9.00, \$6.75, \$5.40, \$4.50 and \$3.86, respectively. Your total bet is \$6, but you will receive only one of these payoffs.

If one of the first three horses wins, you will be ahead. If one of the last three horses wins, you will be behind.

More precisely, to determine the expected amount of money M returned to you, on average, multiply each payoff by the probability it occurs. Assuming the dice are fair, each horse has an equal chance. The probability is 1/6 for each outcome. The amount of money M is $M = \$13.50 \times (1/6) + \$9.00 \times (1/6) + \ldots + \$3.86 \times (1/6) = \$7.17$.

Bet equal amounts; do not assume knowledge of the payoffs.

Your total bet B was \$6. For each dollar bet, your expected payback is $G/B = \$1.195$. Your expected profit per dollar bet is \$0.195 which means you have a 19.5% expected gain per unit bet. This is what is usually meant by the player edge or house edge, as the case may be.

Suppose you object. Say that you would rather bet \$1 on horse one only since it pays the most. Then the pool has \$22; there is a \$2 bet on horse number 1 and it pays 11 for 1. The expected payback on \$1 is $\$11 \times (1/6) + \$0 \times (5/6) = 11/6 = \$1.83$. The profit per \$1 is \$0.83 for an 83% advantage.

Your point would be well taken, but it doesn't work. Since you don't know the payoffs, you don't know which horse to bet.

Does my method have the same flaw? No. I bet equal amounts on each horse. I don't assume any knowledge of the payoffs. The amazing thing is that this method will give you an edge providing: (1) the true winning chances are equal for each horse; and (2) all money in the pool is returned to the players (no track take). In the rare event that the same amount is bet on each horse, you will simply get your money back. *Neither condition (1) nor condition (2) is satisfied at a real race track.*

Why does the method work, when conditions (1) and (2) hold? I'll indicate the proof.

Assume there are n horses. The amounts bet on horses $1, 2 \ldots n$ respectively are $A(1), A(2) \ldots A(n)$. Call the total in the pool $A = A(1) + A(2) + \ldots + A(n)$. The amount returned per dollar bet, in case the winner is horse $1, 2 \ldots n$, respectively, is $A/A(1), A/A(2) \ldots A/A(n)$.

The probability is $1/n$ for each of these returns; the expected payoff for a \$1 bet on each of the n horses is $(A/A(1) + A/A(2) + \ldots + A/A(n)) \times (1/n) = (1/A(1) + \ldots + 1/A(n)) \times A/n$. We have an edge if this exceeds the total amount bet—\$n.

My system works if we can prove for arbitrary non-negative numbers that $(1/A(1) + \ldots + 1/A(n)) \times A/n$ is greater than n (except when $A(1) = A(2) = \ldots = A(n)$).

What if there is a track take? Let K be the fraction of the pool which is returned to the bettors. For simplicity of illustration, this fraction is the same no matter which horse wins. (In practice that isn't necessarily true.)

If the track take is 20%, for example, then $K = 0.8$. If there is no track take, then $K = 1$. Then the formula for your advantage (or disadvantage) E with my system is: $E = (1/A(1) + \ldots + 1/A(n)) \times KA/(n \times n) - 1$.

Let's illustrate the use of this formula with our original example. We had $A(1) = \$2$, $A(2) = \$3 \ldots A(6) = \$7$, $A = \$27$, $n = 6$ horses,

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and $K = 1$ (no track take). Then $E = (1/2 + 1/3 + \ldots + 1/7) \times 27/36 - 1 = 19.5\%$ which agrees with our previous calculation.

Now we ask, "What track take would still allow us to break even in this example?" That's the same as setting the formula for E equal to zero and solving for K. The result is $K = 1/1.19464 = 0.8371$ and $1\text{-}K = 16.29\%$, the amount of the track take which causes us to have zero advantage in this particular betting pool.

If you studied the formula for E, you'd find a pattern. For a given total pool A, the more the bets are unequally distributed among the various horses, the bigger your advantage tends to be. Then you can overcome the larger track take.

What happened in the actual "Love Boat" horse races? On the first race that I recorded, the returns per dollar bet were 3, 3, 3, 3, 4 and 10, respectively. I bet $1 on each horse and learned, when these payoffs were announced, that my expected payback was $(3 + \ldots + 10)/36 = 26/36$. That's an expected loss of $-10/36 = -28\%$. The track take must be huge.

To see how big the track take must be, we use the fact that E is less negative than $K-1$, using my formula for E. Therefore, -28% is less negative than $K-1$ from which the track take is greater than 28%.

In the second race, which I did not bet on, the paybacks per dollar bet were 4, 5, 3, 3, 4, and 5, respectively. Add these to $24 and the track take is more than 33%.

The last race returned on each $2 bet 10, 6, 5, 5, 7 and 10, respectively. The expected payback per dollar bet was 43/72 for a track take of over 40%.

This analysis is probably not exact, because the actual procedure for setting the odds probably differs somewhat from my assumptions. To see why, consider our example in which the bets were 2, 3, ..., 7, respectively, for a total of $27.

Suppose the operators took $6 or 22% from the pool. Then the respective paybacks would be 21/2, 21/3, ..., 21/7, respectively, or $10.50, $7.00, $5.25, $4.20, $3.50, and $3.00. To avoid making change, the operators replace these payoffs by whole dollar amounts. Assume they round down, to make an extra profit ("breakage").

Then the new returns are 10, 7, 5, 4, 3 and 3. These final payoffs don't correspond precisely to any one fixed track take, because of the irregular way in which the breakage can vary from horse to horse.

Next month we'll push this idea further. You'll learn how to cut your disadvantages at jai-alai and at the horse races.

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Dealer's Signature Idea: Another Test of Its Validity

In response to my roulette comments in the September column, reader Allan Wilson wrote in reference to the random number I described. I imply that the number, which is passing when the dealer launches the ball, has no particular relation to the number which came up on the previous spin.

Here's the more precise discussion. Imagine the ball resting in a pocket after a spin. The dealer picks up the ball, pushes the rotor, and launches the ball for the next spin. How many revolutions does the pocket travel beyond the point where the ball is launched for the next spin? This is the pocket in which the ball was last located. The number of revolutions, (called N below), is the important quantity that I call random.

Mathematicians generally use random to mean uncertain. In this sense the number is random. Each random quantity is generally not satisfactorily described unless we know how its values are distributed. The number of revolutions N in question might, for example, be random in that all roulette numbers are equally likely (1/38 probability). That's obviously not true and not what I had in mind.

The number of revolutions N might be fairly well described by a bell shaped curve (the normal probability distribution). If the spread, (standard deviation), in this curve is 17 pockets, the dealer-signature-system can't gain a player advantage, as I discussed previously. That is what I had in mind.

Wilson has a further perceptive observation. He points out a statement made by Kimmel: "The dealer picks up the ball from the previous spin, pushes the wheel just enough to keep it going, and spins the ball around the rim *in one smooth, continuous motion.*"

Wilson states that Kimmel implies that the total action is a regular habitual activity; N will therefore tend to be fairly similar from spin to spin. It won't have as much spread in its values; and one of my objections to the Kimmel system is diminished.

I agree with Wilson. One problem with Kimmel's statement is that the dealer frequently does not conduct just this series of actions. Sometimes he picks up the ball and is diverted briefly which adds a large random increment to N. Sometimes he does give the wheel a push and sometimes he doesn't.

I've watched dozens of roulette dealers, and I recall that frequently this "regular series of action" has an irregularity that alters the value of N considerably.

Here are some counter arguments to these objections about N: (1) Find dealers where N is generally regular, or (2) find a game where you can bet after the ball is spun; then use the number nearest the ball launch point, *not* the last number that came up. This second approach makes the value of N immaterial.

This analysis leads us to a direct experimental test which is quick and easy. It will tell us if the dealer's signature method works for any given dealer.

Kimmel's dealer's signature method I (DSMI). Each time the ball is spun, count the number of relative revolutions that the rotor makes with respect to the ball. Start from the instant the ball is picked up from the rotor.

Do this as follows: Imagine that the ball, when it came to rest after the last spin, left a paint spot in the pocket where it rested. When the ball is picked up, this paint spot rotates away. The next time it passes the ball count one revolution, whether the ball is in the dealer's hand or has already been launched.

Continue to count revolutions each time the paint spot and the ball pass each other. When the ball stops, note what pocket and express the final result as a whole number of revolutions plus the number of pockets. If the ball and paint spot travel W whole revolutions plus p additional pockets, then the total number of revolutions K, (for Kimmel), traveled is $K = W + p/38$ (relative) revolutions.

Let $K_1, K_2, \ldots, K_n$ be the values you record for K on the first spin, second spin, . . ., nth spin. Let $\overline{K} = (K_1 + K_2 + \ldots + K_n)/n$ be the average of these values. Write $\overline{K} = \overline{W} + \overline{p}/38$ where $\overline{W}$ is the whole number part of $\overline{K}$. Then solve for $\overline{p}$.

The result is the estimated average number of pockets. To bet, take the last pocket to come-up and move forward $\overline{p}$ pockets to predict the next number.

Will it give you an edge? It will if the spread of K around $\overline{p}$ is small enough. How small? Less than 17 pockets if K is normally distributed. How do we find the size of the spread? Calculate the square root of $[(K_1 - \overline{K})^2 + \ldots + (K_n - \overline{K})^2]/(n-1)$ and call the result "S" (spread).

This is the estimate of the spread from the data. The key to the experimental test of the Kimmel method is to find a dealer, if you can, where S is small enough. (I invite you to send data to me. I'll report the results in this column.)

Technical note: If the actual distribution of K values differ somewhat from a normal distribution, then the critical value of S, required for a player advantage, will also vary slightly. For some types of K distribution, a true S of, say 19 or 20, or a somewhat larger data-based S, might be consistent with a player advantage. For other types of K distribution, a true value for S of 14, 15 or a somewhat smaller data-based S, might not give a player advantage.

If S turns out to be between, say, 13 and 21 for $n=200$, we probably don't have a clear cut decision unless we investigate in further detail. If S is less than 13, I think the Kimmel system will almost surely work. If S is greater than 21, I think the Kimmel system won't work.

I expect S to be considerably greater than 21, but I won't feel that I'm right until I've checked it with data on K. After several dealers are tested, we'll have an idea of how S varies from one dealer to another. We will have an idea of whether we're likely to find any good dealers, and if so how frequently.

A variation on DSM I is DSM II.

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Here we imagine that just as the dealer launches the ball, a paint spot appears on the pocket that is closest to the launch point. This pocket and paint spot play the DSM I roles. Instead of counting from the last winning number, we are counting from the pocket closest to the launch point of the ball.

This method has the advantage that N is eliminated from the method. The uncertainties from it are eliminated too. Thus, the method seems more likely to work.

It has disadvantages. First, there is some uncertainty in deciding just which number is closest to the ball when it is launched. This uncertainty itself introduces an error. Second, you have to wait until the ball is launched to place your bet. Many roulette games I've seen, especially in Europe, don't allow this. **gt**

Trackside East

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fast track condition. As a result, the front runners got the edge. Bettors were constantly moaning and saying play the speed until it rains or the track surface is slowed for some reason. Who said bettors don't know all the angles?

Two systems at Saratoga, (maybe not entirely new), the daily double and an exacta spinoff, were favorites. Take the top two favorite selections from Associated Press and *Daily Racing Form* and hook them up with the top two in the second. It worked better in the late double, because of the eighth heavy favorite feature race. Exactas were nothing more than taking the top horse and criss-crossing him with second and third choices. An \$8 dollar wager and this method really works.

Saratoga is a track full of betting surprises and one where new racing stars are created! **gt**