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The Mathematics of Gambling

The Kelly Money Management System

by Edward O. Thorp

The question I will answer is how to "manage your money" in betting or investment situations. There is, in fact, a rule or formula which you can use to decide how much to bet. I will explain the rule and tell you what benefits are likely if you follow it.

Let's begin with a simple illustration that I deliberately exaggerated to better get the idea across. Suppose you have a very rich adversary who will let you bet any amount on heads at each toss of a coin and that you both know that the chance of heads is some number p greater than 1/2. If your bet pays even money, then you have an edge. Now suppose $p=0.52$, so you tend to win 52 percent of your bets and lose 48 percent. This is similar to the situation in blackjack when the ten-count ratio is about 1.5 percent. Suppose too that your bankroll is only $200. How much should you bet? You could play safe and just bet one cent each time. That way, you would have virtually no chance of ever losing your $200 and being put out of the game. But your expected gain is .04 per unit or .04 cents per bet. At 100 one cent bets an hour, you expect to win four cents per hour. It's hardly worth playing.

Now look at the other extreme where you bet your whole bankroll. Your expected gain is $4 on the first bet, more than if you bet any lesser amount. If you win, you now have $200. If you again bet all of it on your second turn, your expected gain is $8 and is more than if you bet any lesser amount. You make your expected gain the biggest on each turn by betting everything. But if you lose once, you are broke and out of the game. After many turns, say 20, you have won 20 straight tosses with probability $.52^{20}=0.000002090$ and have a fortune of $104,857,600, or you have lost once with probability 0.999997910 and have nothing. In general, as the number of tosses n increases, the probability that you will be ruined tends to 1 or certainty. This makes the strategy of betting everything unattractive.

Since the gambling probabilities and payoffs at each bet are the same, it seems reasonable to expect that the "best" strategy will always involve betting the same fraction of your bankroll at each turn. But what fraction should this be? The "answer" is to bet $p-(1-p)=0.52-0.48=0.04$, or four percent of your bankroll each time. Thus you bet $4 the first time. If you win, you have $104, so you bet $0.04 \times \$104=\$4.16$ on the second turn. If you lost the first turn, you have $96, so you bet $0.04 \times \$96=\$3.84$ on the second turn. You continue to bet four percent of your bankroll at each turn. This strategy of "investing" four percent of your bankroll at each trial and holding the remainder in cash is known in investment circles as the "optimal geometric growth portfolio" or OGGP. In the 1962 edition of *Beat the Dealer*, I discussed its application to blackjack at some length. There I called it the Kelly system, after one of the mathematicians who studied it, and I also referred to it as (optimal) fixed fraction (of your bankroll) betting.

Why is the Kelly system good? First, the chance of ruin is "small." In fact, if money were infinitely divisible (which it can be if we use bookkeeping instead of actual coins and bills, or if we use precious metals such as gold or silver), then any system where you never bet everything will have zero chance of ruin because even if you always lose, you still have something left after each bet. The Kelly system has this feature. Of course, in actual practice coins, bills or chips are generally used, and there is a minimum size bet. Therefore, with a very unlucky series of bets, one could eventually have so little left that he has to bet more of his bankroll than the system calls for. For instance, if the minimum bet were $1, then in our coin example, you must overbet once your bankroll is below $25. If the minimum bet were one cent, then you only have to overbet once your bankroll falls below 25 cents. If the bad luck then continues, you could be wiped out.

The second desirable property of the Kelly system is that if someone with a significantly different money management system bets on the same game, your total bankroll will probably grow faster than his. In fact, as the game continues indefinitely, your bankroll will tend to exceed his by any preassigned multiple.

The third desirable property of the Kelly system is that you tend to reach a specified level of winnings in the least average time. For example, suppose you are a winning card counter at blackjack, and you want to run your $400 bankroll up to $40,000. The number of hands you'll have to play on average to do this will, using the Kelly system, be very close to the minimum possible using any system of money management.

To summarize, the Kelly system is relatively safe, you tend to have more profit, and you tend to get to your goal in the shortest time.

Blackjack Money Management

The Kelly system calls for no bet unless you have the advantage. Therefore, it would tell you to avoid games such as craps and keno and slot machines. However, if you have the knowledge and skill to gain an edge in blackjack, you can use the Kelly system to optimize your rate of gain. The situation in blackjack is more complex than the coin toss game because (1) the payoff on a one-unit initial bet can vary widely, due to such things as dealer or player blackjacks, insurance, doubling down, pair splitting, and surrender, and (2) because the advantage or disadvantage to the player varies from hand to hand.

However, we can apply the coin toss results to blackjack by making some slight modifications. First, let's see where the coin toss example's best fixed fraction of four percent came from. The general mathematical formula for the Kelly system is this: In any (single) favorable gambling situation or investment, bet that fraction f of your bankroll which maximizes $E\ ln\ (1 + f)$, where E is the expected value and ln is the natural logarithm (to the base $e = 2.71828\ldots$). This ln function is available on most hand calculators. In the case of coin tossing, the best f, which I call f^*, is given for a favorable bet by $f^* = 2p - 1$, where p is the chance of success on one toss, and $f^* = 0$ if $p \leq 1/2$, i.e., if the game is either fair or to your disadvantage. Note too that $f^* = 2p - 1$ is coincidentally your expected gain per unit bet.

Now your expected gain in blackjack varies from hand to hand. If we think of successive hands as coin tosses with a varying p, then we should bet $f^* = 2p - 1$ whenever our card count shows that the deck is favorable. When the deck is unfavorable, we "should" bet zero. Uston-type team play approximates this ideal of betting zero in unfavorable situations. You can also achieve this sometimes by counting the deck and waiting until the deck is favorable before placing your first bet. But it is impractical to bet zero in unfavorable situations, so we bet as small as is discreet. Think of these smaller, slightly unfavorable bets as a "drain" or "tax" which "water down" the overall advantage of the favorable bets. To compensate for this reduced advantage, f^* should generally be "slightly" smaller than the $2p - 1$ computed above. Another effect of the small, slightly unfavorable bets is to increase the chance of ruin a little.

The most important blackjack "correction" to the f^* computed for coin tossing is due to the greater variability of payoff. Peter Griffin calculates that the "root mean square" payoff on a one-unit blackjack bet is about 1.13. It turns out then that f^* should be corrected to about $(2p - 1)/1.27$ or about .79 times the advantage. Shade this to .75 because of the "drain" of the small, unfavorable bets and we have the fairly accurate rule: For favorable situations at blackjack, it is (Kelly) optimal to bet a percent of your bankroll equal to about 3/4 of your percent advantage. For instance, with a $400 bankroll and a one percent advantage, bet 3/4 of one percent of $400, or $3.

The Kelly System for Roulette

In general in roulette, the house has the edge, and the Kelly system says, "don't bet." But in my series on physical prediction at roulette, I will describe a method where we (Shannon and I), with the aid of an electronic device, had an edge of approximately 44 percent on the most favored single number. That corresponds to a win probability of $p = 0.04$, with a payoff of 35 times the bet, and a probablilty of $1 - p = 0.96$ of losing the bet. It turns out that $f^* = .44/35 \doteq .01257$. The general formula for f^* when you win A times a favorable bet with probability p and lose the bet with probability $1 - p$, is $f^* = e/A$ where $e = (A + 1)\,p - 1 > 0$ is the player's expected gain per unit bet or his advantage. Here $A = 35$, $p = .04$, and $e = 0.44$. In the coin toss example, $A = 1$, $p = .52$, and $e = .04$.

Using any fixed betting function f, the "growth rate" of your fortune is $G(f) = p\ ln\ (1 + Af) + (1 - p)\ ln\ (1 - f)$. After N bets, you will have approximately $exp\{N\ G(f)\}$ times as much money, where exp is the exponential function, also given on most pocket calculators.

For the roulette single number example, using my hand calculator (an HP65) gives $G(f^*) = 0.04\ ln\ (1 + 35\ f^*) + 0.96\ ln\ (1 - f^*) = .04\ ln\ (1.44) + 0.96\ ln\ (0.98743) = .04 \times .36464 + 0.96 \times (-0.01265) = 0.1459 - .01215 = .00244$. After 1,000 bets, you will have approximately $exp\{2.44\} = 11.47$ times your starting bankroll.

Notice the small value of f^*. That's because the very high risk of loss on each bet makes it too dangerous to bet a large fraction of your bankroll. To show the advantages of diversification, suppose instead that we divide our bet equally among the five most favored numbers, as Shannon and I actually did in the casinos. If one of these numbers comes up, we win an amount equal to $(35 - 4)/5$ of our bet, and if none come up, we lose our bet. Thus $A = 31/5 = 6.2$. The other four numbers are not quite as favored as the best number. However, to illustrate diversification, suppose that the five-way bet has the same .44 advantage. This corresponds to $p = 0.20$. Then $f^* = .44/6.2 = 0.07097$, so you bet about seven percent of your bankroll and $G(f^*) = 0.20\ ln\ (1 + 6.2f^*) + 0.80\ ln\ (1 - f^*) = 0.01404$. This growth rate is about 5.75 times that for the single number. After 1,000 bets, you would have approximately 1.25 million times your starting bankroll. Such is the power of diversification.

What is the price of deviating from betting the optimal Kelly fraction f^*? I will discuss this in a subsequent column. It turns out that for bet payoffs like blackjack, which can be approximated by coin tossing, the "performance loss" is not serious over several days play. But for the roulette example, the performance loss from moderate deviations from the Kelly system is considerable.

ASK OUR EXPERTS

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allowed is to bet three units on the pass line. I don't care what your units are. They can be three $1 chips, three $5 chips or three $25 chips, or any kind of bet that can be divided into three units. For example, a $150 bet is three units of $50 each. The reason I suggest a three-unit bet is this: If you bet any three units on the pass line, if the point is 6 or 8, you can bet five units behind the line as a free odds bet. If the point is 5 or 9, you can bet four units behind the line as a free odds bet. In both instances, you have a little extra edge on the bet. The same holds true for come bets.

Q: *I'm planning a trip to Lake Tahoe, and I know there are several casinos up there. I've been studying up on blackjack and also reading the articles in your magazine by Stan Roberts and am anxious to play. But I'd like to get into a sin-*

gle deck game. Are there any single deck games in Lake Tahoe?

A.M.
Eugene, OR

A: Practically every blackjack game dealt in Lake Tahoe is a single deck game, so you'll have no trouble at all playing single deck. If you've paid attention to the articles by Stan Roberts, you should have no trouble beating the game in Lake Tahoe.

Q: *If you go on a junket to a Las Vegas hotel, must you do all your gambling in their casino? I'm signed up for a trip to Vegas to the ———— Hotel, and I want to know what I have to do about gambling there. Can I go to another casino and do some playing?*

B.L.T.
Detroit, MI

A: The hotel you mentioned will expect you to give them a minimum of $7,500 worth of action at their tables. In other words, that's how much in markers you'll be expected to take out while at their tables. You just can't sign markers (IOUs) and then play for a few minutes and expect the casino to comp you to all the free services, including air fare, that is given to junket members. After you get the chips, you'll have to do some serious playing at the table.

Remember, you have to play $7,500 worth, not lose that much. The casino would prefer that you lose, but even if you win, you'll have fulfilled your requirements if you've given them legitimate play at their tables.

As to playing in other casinos, why not? The hotel won't care, so long as you give them the minimum play they expect from you.

Q: *I know you've had some articles in the magazine about bridge. I don't know too much about the game, just starting to play, but some players were talking about a Yarboro. Is that the correct spelling? And what is it?*

T.M.
Riverside, CA

A: You've spelled it incorrectly. It's Yarborough, and it means an absolute bust hand, with no cards above a 10 and all the suits equally divided. In other words, a useless hand for bidding and playing. **9**

"That's right, play nice in a true spirit of Christmas."