The Mathematics of Gambling

Systems for Roulette II

by Edward O. Thorp

One reason I chose roulette to illustrate mathematical systems is that it is easy to understand the odds and probabilities. For this discussion I assume that each of the 38 numbers is equally likely to come up on any spin. There are 18 Red numbers so we define the *probability* that Red comes up on any spin to be 18/38, or the number of ways for Red to come up divided by the total number of outcomes. Probabilities for other roulette bets are figured similarly. For instance, the probability that the "first dozen," meaning one of the numbers from 1 to 12, occurs is 12/38. The *player expectation* on any bet in any game is computed by multiplying each possible gain or loss by the probability of that gain or loss, then adding the two figures. Expectation is the amount you *tend* to gain or lose on average when you bet, so it is a very important idea. For instance, the player expectation for a $7 bet on Red is (18/38)x($7)+(20/38)x(−$7) or −$14/38 or about −$.37 where 20/38 is the probablility Red does not come up and −$7 is the amount lost in that case.

As another example, suppose that on the first hand of four-deck blackjack the player bets $12, he is dealt 6,5, and the dealer then shows an ace up. The dealer asks the player if he wants insurance. This is a separate $6 bet. It pays $12 if the dealer's hole card is a ten-value. It pays −$6 otherwise. A full four-deck pack has 64 tens and 144 non-tens. Three non-tens are gone, so of the 205 unseen cards, 141 are non-tens. *Assuming* the deck is "randomly" shuffled (this means that all orderings of the cards are equally likely), the chances are equally likely that each of these 205 unseen cards is the dealer's hole card. Thus the player's expectation is (64/205)x($12)+(141/205)x(−$6)=−$78/205 or about −$.38. The player should not take insurance.

Different betting amounts have different expectations. But the player's expectation as a *percent of the amount bet* is always the same number. In the case of betting on the Red in roulette, this is 18/38 −20/38=−2/38=−1/19 or about −5.26%. Thus, the expectation of any size bet on Red at American double-zero roulette is −1/19 or about −5.26% of the total amount bet. So to get the expectation for any size bet on Red, just multiply by −5.26%. With one exception, the other American double-zero roulette bets also have this expectation per unit bet. The player's expectation per unit is often simply called the player's disadvantage. What the player loses, the house wins, so the *house advantage, house percentage,* or house expectation per unit bet by the player is +5.26%.

A useful basic fact about the player's expectation is this: the expectation for a series of bets is the total of the expectations for the individual bets. For instance, if you bet $1 on Red, then $2, then $4, your expectations are −$2/38, −$4/38, and −$8/38. Your total expectation is −$14/38 or (a loss of) about −$.37. Thus, if your expectation on each of a series of bets is −5.26% of the amount bet, then the expectation on the whole series is −5.26% of the total of all bets. This is one of the fundamental reasons why "staking systems" don't work: a series of negative expectation bets must have negative expectation.

One correct version of the so-called "law of averages" says that in a "long" series of bets, you will *tend* to gain or lose "about" the total expectation of those bets. This means that a series of "bad" bets is also "bad," and that systems don't help.

Applying these ideas to the doubling-up system, let's calculate the player's expectation for one *cycle.* Think of a complete cycle as a single (complicated looking) bet. Now refer to Table 2 of last month's column. The fifth column gives the probability that the cycle ends on turn #1, #2, etc. and the fourth column gives the gain or loss for each of these cases. Multiply each entry in the fourth column by the corresponding entry in the fifth column. Then add the results: $\$1\text{x}18/38+\$1\text{x}20/38\text{x}18/38+\ldots+\$1\text{x}(20/38)^{9}\text{x}18/38-\$23\text{x}(20/38)^{10}\text{x}18/38-\$2023\text{x}(20/38)^{11}$ which simplifies to $1-24\text{x}(20/38)^{10}-2000\text{x}(20/38)^{11}=1-0.0391\ldots-1.7168\ldots=-\$.7560266578\ldots$ . . . Thus, the expected loss to the bettor is about −$.76 per cycle.

Now let's calculate the expected (or "average") amount bet on one cycle. Referring again to last month's Table 2, we see that if the cycle ends on turn #1, the total of all bets is $1, if it ends on turn #2, the total of all bets is $1+$2, if it ends on turn #3, the total is $1+$2+$4, etc. If the cycle ends on turn #11, the total amount bet is $2,023. (To get these totals as of the end of any turn, add columns two and three.) Then multiply these total amounts bet by the chances in column 5 to get $\$1\text{x}18/38+\$2\text{x}(20/38)\text{x}(18/38)+\$4\text{x}(20/38)^{2}\text{x}(18/38)+\ldots+\$512\text{x}(20/38)^{9}\text{x}(18/38)+\$2023\text{x}(20/38)^{10}\text{x}(18/38)+\$2023\text{x}(20/38)^{11}$ which simplifies to $\$2\text{x}(18/38)\text{x}((40/38)^{10}-1)/(40/38-1)+\$2024\text{x}(20/38)^{10}-\$1=\$14.3645065$. If we divide the expected loss by the average bet per cycle we get $-\$.756\ldots \div \$14.36\ldots = -1/19$ *exactly* or -5.26%.

These calculations are tedious, and for each system the details are different, so they have to be done again. And there are an infinite number of gambling systems, so calculations can never check them all anyhow. Clearly this is not the way to understand gambling systems. The correct way is to develop a general mathematical theory to cover gambling systems. That has been done and here's how it works. First, we define the *action* in a specified set of bets to be the total of all bets made. From what we said above, your expected (gain or) loss is your action (i.e. the total of all your bets) times the house edge. For example, if you bet $10 per hand at blackjack and play for 10 hours, betting 100 hands per hour, you have made a thousand $10 bets, which is $10,000 worth of "action." If you are a poor blackjack player and the casino has a 3% edge over you, your expected loss is $\$10{,}000 \times 3\% = \$300$. Your actual loss may be somewhat more or somewhat less.

If Nevada casino blackjack grosses a total of $400 million per year and the average casino edge over the player is 2% of the initial wager, then we can determine the total action (A) per year: $.02A = \$400{,}000{,}000$ so $A = \$20$ billion. Thus from these figures we would estimate $20 billion worth of bets are made per year at Nevada blackjack. The 2% figure might be substantially off. We could get a fairly accurate idea of the true figure by making a careful statistical sampling survey. If, instead, the figure is 4%, then $A = \$10$ billion. With 1%, $A = \$40$ billion per year.

Using the concept of *action*, we can now understand the famous "law of averages." This says, roughly, that if you make a *long* series of bets and record both the action (A) and your total profit or loss (T), then the *fraction* T/A is approximately the same as the *fraction* E/A where E is the total of the *expected* gain or loss for each bet. Many people misunderstand this "law." They think that it says the E and T are approximately the same after a long series of bets. This is false. In fact, the difference between E and T tends to get larger as A gets bigger.

For example, suppose you are matching coins and betting $1 each time. The chances are equal to win or lose on each bet. Each bet has zero expectation: $\$1 \times 1/2 - \$1 \times 1/2 = 0$. Therefore $E=0$ for any series of bets. The action equals $\$1 \times N$ where N is the number of bets. But your total profit or loss can be shown to have an average deviation from E of about $\sqrt{N}$. Let $D = T - E$ be the difference or deviation between what you actually gain or lose (T), and the expected gain or loss (E). Therefore, for 100 bets, the average deviation from $E=0$ is about $10 (in fact, the chances are about 68% that you'll be within $10 of even; they're about 98% that you'll be within $20 of even). For ten thousand $1 bets it's about $100 and for a million $1 bets it's about $1,000. Table 1 shows what happens. For instance, the last line of Table 1 says that if we match coins one million times at $1 per bet, our expected gain or loss is zero (a "fair" game). But on average, we'll be about $1,000 ahead or behind. In fact, we'll be between +$1,000 and −$1,000 about 68% of the time. (For a million $1 bets, the deviation D has approximately a normal probability distribution with mean zero and standard deviation $1,000.) However (fifth column) $D/A = 0.001$, so the deviation as a percent of the action is very small. And about 68% of the time T/A is

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TABLE 1

number of bets N (A = $N)	expected gain E	average size of D is about $\sqrt{N}$	about 68% of time T is between	average size of D/A	about 68% of time T/A is between
1	0	1	1 and - 1	1	1 and -1
100	0	10	10 and - 10	0.1	0.1 and -0.1
10,000	0	100	100 and - 100	0.01	0.01 and -0.01
1,000,000	0	1000	1000 and -1000	0.001	0.001 and -0.001

TABLE 2

N (A = $N)	E	$D \sim \sqrt{N}$	about 68% of time T is between	$D/A \sim 1/\sqrt{N}$	about 68% of time T/A is between
1	- $.05	1	+ $.95 and - $1.05	1	+.95 and -1.05
100	- $5.26	10	+ $4.74 and - $15.26	0.1	+.0474 and -.1526
10,000	- $526.00	100	- $426.00 and - $626.00	0.01	-.0426 and -.0626
1,000,000	-$52631.00	1,000	-$51631.00 and -$53631.00	0.001	-.0516 and -.0536

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between —.001 and +.001. so *as a percent of the action* the result is very near the expected result of zero. Note that the average size of D, the deviation from the expected result E, grows—contrary to popular belief. However, the average size of the percentage of deviation, D/A, tends to zero, in agreement with a correct version of the "law of averages."

For $1 bets on Red at American roulette, the corresponding results appear in Table 2. Notice that in the last column the spread in T/A gets closer and closer to E/A= —.0526. This is where we get the statement that if you play a "long time" you'll lose about 5.26% *of the total action*. Note, too, in column 4 that there appears to be less and less chance of being ahead as the number of trials goes on. In fact, it can be shown that in all negative expectation games the chance of being ahead tends to zero as play continues.

Now, the ordinary player probably won't make a million $1 bets. But the casino probably will see that many and more. From the casino's point of view, it doesn't matter whether one player makes all the bets or whether a series of players does. In either case, its profit in the long run is assured and will be very close to 5.26% of the action. With many players, each making some of the 1,000,000 bets, some may be lucky and win, but these will generally be compensated for by others who lose more than the expected amount. For instance, if each of 10,000 players take turns making a hundred $1 bets, Table 2 tells us that about 68% of the time their result will be between +$4.74 and —$15.26. About 16% of the time the player wins more than $4.74 ("lucky") and about 16% of the time the player loses more than $15.26 ("unlucky"). But players cannot predict or control which group they'll be in.

This same "law of averages" applies to more complicated sequences of bets. For instance, suppose you bet $10 on Red at roulette (E=—$.53), then bet $100 on "players" at Baccarat (E= —$1.06), then bet $10 on a hand in a single-deck blackjack game where the ten-count is 15 tens, 15 others (E=+$.90). The total E is —$.53—$1.06+$.90=—$.69. The total A is $10+$100+$10=$120. If you make a long series of bets and Record E and A as well as your gains and losses for each one, then just as in the coin matching example (Table 1) and the roulette example (Table 2), the fraction D/A tends to zero so T/A tends to E/A. That means that over, say, a lifetime, your total losses as a *percent* of your total action will tend to be very close to your total expectation as a percent of your action.

If you want a good gambling life, make positive expectation bets. You can, as a first approximation, think of each negative expectation bet as charging your account with a tax in the amount of the expectation. Conversely, each positive expectation bet might be thought of as crediting your account with a profit in the amount of the expectation. If you only pay tax, you go broke. If you only collect credits, you get rich.

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have heard Entratter laugh with much relief when she told him she was just pulling his leg.

Since then she and Roger Smith have been denying rumors of impending motherhood almost six times a year to newsmen. And Ann-Margret keeps telling the news boys that she and Roger are just not ready for her to start having babies. One major reason is that she's an absolute "workaholic" and just can't take the time for such domestic bliss. And no wonder. As one of the stars of Caesars Palace she earns a million dollars yearly in a pact that has three more years to run. She still has an action loaded act that has her dancing and singing her little Swedish head off.

On the subject of babies, she told me recently backstage at Caesars Palace, "When the time comes, I promise I'll let you know before I tell any other newsman. As a matter of fact, I'll let you know before I even tell Roger." Now that's what I call being a nice lady.