SOME BANACH SPACES CONGRUENT TO THEIR CONJUGATES

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Mazur remarked that there are separable infinite dimensional Banach spaces which are congruent (*i.e.* isometrically isomorphic) with their conjugates [**1**; page 245]. I am not aware of the reference to the example(s) Mazur had in mind. In [**3**; page 195] the term "self-conjugate space" is introduced but not defined, it merely being noted that $L_2(0, 1)$ is self-conjugate because it is congruent to its conjugate and that (notation from this point follows [**2**]) l_2 and l_2^n are also self-conjugate.

This suggests considering the general question "Which Banach spaces are congruent to their conjugate?". Theorem 1 describes some of these B-spaces. Congruence of spaces will be denoted by equality.

Theorem 1. *Let $(X, |.|)$ be any B-space congruent to X^{**}. Let $\|.\|$ be a norm for the real plane R^2 which coincides with some positive scalar multiple c of the Euclidean norm on the non-negative quadrant. Let $Z = (X \oplus X^*, |\|.\||)$, where $|\|(x, x^*)\|| = \|(|x|, |x^*|)\|$. Then Z and Z^* are congruent.*

Proof. Since $Z^* = (X \oplus X^*)^*$ and $X^* \oplus X^{**} = X^* \oplus X = X \oplus X^* = Z$, it suffices to show that $(X \oplus X^*)^* = X^* \oplus X^{**}$.

Given $z^* \in Z^*$, we define $z_1^* \in X^*$, $z_2^* \in X^{**}$ by $z_1^*(x) = z^*\big((x, 0)\big)$, $z_2^*(x^*) = z^*\big((0, x^*)\big)$, and consider the mapping ϕ of Z^* onto $X^* \oplus X^{**}$ given by $\phi(z^*) = (z_1^*, z_2^*)$. It is easy to verify that ϕ is a linear isomorphism onto. It therefore suffices to show further that it is an isometry.

We have

$$\begin{aligned}
|z^*| &= \sup\{|z^*(x, x^*)| : \|(|x|, |x^*|)\| \leqslant 1\} \\
&= \sup\{|z^*(x, 0)| + |z^*(0, x^*)| : \|(|x|, |x^*|)\| \leqslant 1\} \\
&= \sup\{|z_1^*(x)| + |z_2^*(x^*)| : \|(|x|, |x^*|)\| \leqslant 1\} \\
&= \sup\{|z_1^*|\,|x| + |z_2^*|\,|x^*| : \|(|x|, |x^*|)\| \leqslant 1\} \\
&= \|(|z_1^*|, |z_2^*|)\|' \\
&= \|(|z_1^*|, |z_2^*|)\|,
\end{aligned}$$

where we let $(R^2, \|.\|')$ be the normed conjugate of $(R^2, \|.\|)$.

To see the second equality, note that $\leqslant$ is evident. But each term on the right appears on the left if we replace x by $x \operatorname{sgn}\big(z^*(x, 0)\big)$ and x^* by $x^* \operatorname{sgn}\big(z^*(0, x^*)\big)$ where for any scalar $re^{i\theta}$, $\operatorname{sgn} re^{i\theta} = e^{-i\theta}$ if $r \neq 0$ and $\operatorname{sgn} re^{i\theta} = 0$ if $r = 0$. Hence $\geqslant$ holds also, so we have equality.

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In the fourth equality, $\leqslant$ is again evident. To see that $\geqslant$ holds, suppose $\epsilon > 0$ is given. Then there are vectors x_0 and x_0^* such that $|z_1^* x_0| > (|z_1^*| - \epsilon)|x_0|$ and $|z_2^* x_0^*| > (|x_2^*| - \epsilon)|x_0^*|$ and hence for $|x_0|$ and $|x_0^*| \leqslant c$, in particular when $\|(|x_0|, |x_0^*|)\| \leqslant 1$, we have

$$|z_1^* x_0| > |z_1^*|\,|x_0| - c\epsilon, \quad |z_2^* x_0^*| > |x_2^*|\,|x_0^*| - c\epsilon.$$

(*) Therefore $\quad |z_1^* x_0| + |z_2^* x_0^*| > |z_1^*|\,|x_0| + |z_2^*|\,|x_0^*| - 2c\epsilon.$

Now

$$\sup_{\|(|x|, |x^*|)\| \leqslant 1} (|z_1^*|\,|x| + |z_2^*|\,|x^*|) = \sup_{\|(a, b)\| \leqslant 1} (a|z_1^*| + b|z_2^*|)$$

where a and b are non-negative scalars. But if we replace x_0 by $ax_0/|x_0|$ and x_0^* by $bx_0^*/|x_0^*|$, we see that for any fixed x_0 and x_0^*,

$$\sup_{\|(|x_0|, |x_0^*|)\| \leqslant 1} (|z_1^*|\,|x_0| + |z_2^*|\,|x_0^*|) = \sup_{\|(a, b)\| \leqslant 1} (a|z_1^*| + b|z_2^*|).$$

Hence by taking suprema of each side of (*), we find that the left side of the fourth equality is $\geqslant$ the right side $-2c\epsilon$. Since ϵ, and hence $c\epsilon$, is arbitrary, the left side is $\geqslant$ the right side, so the desired equality is established.

The fifth equality follows from the definition of normed conjugate.

The sixth equality is evident for positive multiples of the Euclidean norm. Lemma 2, below, establishes it for the more general norms hypothesised in the theorem.

LEMMA 2. *Let $\|.\|$ be a norm for R^2 which concides with a positive scalar multiple c of the Euclidean norm on the non-negative quadrant. If $(R^2, \|.\|')$ is the normed conjugate of $(R^2, \|.\|)$ then $\|.\|'$ also coincides with a positive scalar multiple of the Euclidean norm on the non-negative quadrant.*

Proof. It suffices to consider the case $c = 1$. In this case the unit sphere is a subset of $\{(a, b) : \max(|a|, |b|) \leqslant 1\}$. If not, there is an (a, b) such that $\|(a, b)\| \leqslant 1$ and either $|a|$ or $|b| > 1$. We may suppose without loss of generality that $b > 1$, $a < 0$. Then any segment joining (a, b) to a point of norm 1 in the *positive* quadrant which is sufficiently close to $(0, 1)$ "passes over" the point $(0, 1)$. But each point on the segment has norm less than or equal to 1, contradicting the fact that $\|(0, 1)\| = 1$.

Thus the supremum of any functional in the non-negative quadrant of $(R^2, \|.\|')$ is attained on that part of the unit sphere of $(R^2, \|.\|)$ lying in the non-negative quadrant, hence $\|.\|'$ coincides with the Euclidean norm in the non-negative quadrant.

Remark. There are as many norms $\|.\|$ for R^2 of the type described in the lemma as there are convex monotone functions joining $(-1, 0)$ and $(0, 1)$.

One might be tempted to conjecture that Theorem 1 is valid whenever $(R^2, \|.\|) = (R^2, \|.\|')$, for example when $\|(a, b)\| = |a| + |b| = |.|_1$ and

$\|(a, b)\|' = \max(|a|, |b|) = |.|_\infty$. The next lemma disproves this conjecture.

LEMMA 3. *The spaces*

$$(l_1^n \oplus l_\infty^n, |.|_1)^* = (l_1^n \oplus l_\infty^n, |.|_\infty) \text{ and } (l_1^n \oplus l_\infty^n, |.|_1),$$

taken over the real scalar field, are not congruent when $n \geqslant 2$.

Proof. The spaces are congruent if and only if the sixth equality in the proof of Theorem 1 holds. Thus the problem is equivalent to showing that $(l_1^n \oplus l_\infty^n, |.|_\infty)$ is not congruent to $(l_1^n \oplus l_\infty^n, |.|_1)$.

Congruence preserves extreme points, so it will suffice to show that $(l_1^n \oplus l_\infty^n, |.|_\infty)$ has $2n \times 2^n$ extreme points while $(l_1^n \oplus l_\infty^n, |.|_1)$ only has $2n + 2^n$ extreme points, because $2n \times 2^n > 2n + 2^n$ for $n \geqslant 2$.

Denote the unit vectors in l_1^n by e_i, $1 \leqslant i \leqslant n$, and in l_∞^n by e_i, $n+1 \leqslant i \leqslant 2n$. We assert that the extreme points in $(l_1^n \oplus l_\infty^n, |.|_1)$ are precisely the $2n$ vectors $\pm e_i$, $1 \leqslant i \leqslant n$ and the 2^n vectors $\pm e_{n+1} \pm \cdots \pm e_{2n}$ which can be obtained by arbitrary choice of $\pm$.

It suffices to consider points x of norm 1 and test segments with end-points y, z of norm 1. Let $\sum_{i=1}^{n} |x_i| + \max_{n+1 \leqslant i \leqslant 2n} |x_i| = 1$, and similarly for $y = (y_i)$ and $z = (z_i)$, and suppose $x_i = a y_i + (1-a) z_i$ for $1 \leqslant i \leqslant 2n$. If x is extreme, $|x_i| = c$, $n+1 \leqslant 1 \leqslant 2n$, is easily seen to be necessary. Further, either $c = 0$ or $c = 1$. If not, then $0 < \sum_{i=1} |x_i| = c < 1$ and for some i_0 such that $1 \leqslant i_0 \leqslant n$, $|x_{i_0}| = d > 0$. Then if Δx is the vector defined by $(\Delta x)_{i_0} = \big(\min(1-c, c)\big) x_{i_0}$; $x_i = 0$, $1 \leqslant i \leqslant n$ and $i \neq i_0$; $(\Delta x)_i = \big(\min(1-c, c)\big) x_i$, $n+1 \leqslant i \leqslant 2n$; then $x = (x+\Delta x)/2 + (x - \Delta x)/2$ where $|x \pm \Delta x| = 1$.

Two cases remain. They are $\sum_{i=1}^{n} |x_i| = 1$, which is extreme if and only if $x = e_i$, $1 \leqslant i \leqslant n$; and $|x_i| = 1$, $n+1 \leqslant i \leqslant n$, which is extreme if and only if $x = \pm e_{n+1} \pm \cdots \pm e_{2n}$, where the $\pm$ signs may be chosen arbitrarily.

We further assert that the extreme points in $(l_1^n \oplus l_\infty^n, |.|_\infty)$ are precisely the $2n \times 2^n$ vectors of the form $\pm e_i \pm e_{n+1} \pm \ldots \pm e_{2n}$, where $1 \leqslant i \leqslant n$, which can be obtained by arbitrary choices of $\pm$.

The proof of the last statement is similar to the previous proof and so it has been omitted.

References

1. S. Banach, *Théorie des opérations linéaires* (Chelsea, New York, 1955).
2. N. Dunford and J. Schwartz, *Linear operators*, Vol. I (Interscience, New York, 1958).
3. L. A. Lusternik and V. J. Sobolev, *Elements of functional analysis* (Gordon and Breach, New York, 1961).

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