PROJECTIONS ONTO THE SUBSPACE OF COMPACT OPERATORS

E. O. THORP

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Introduction. The purpose of this paper is to establish the following theorem.

THEOREM. *Suppose U and V are Banach spaces and that there are bounded projections P_1 from U onto X and P_2 from V onto Y. Then there are no bounded projections from the space of bounded operators on U into V onto the closed subspace of compact operators, in the following cases:*

1. *X is isomorphic* [1] *to $\ell^p, 1 \leq p < \infty$; Y is isomorphic to ℓ^q, $1 \leq p \leq q \leq \infty$ or c_0 or c.*
2. *X is isomorphic to c_0; Y is isomorphic to ℓ^∞, c_0 or c.*
3. *X is isomorphic to c; Y is isomorphic to ℓ^∞.*

NOTATION. If X and Y are Banach spaces, $[X, Y]$ is the set of bounded linear operators from X into Y. ℓ^∞ is the set of bounded sequences with the sup norm.

A space X is said to have a countable basis if there is a countable subset of elements of X, called a basis, such that each $x \in X$ is uniquely expressible as

$$x = \sum_{i=1}^{\infty} \xi_i \varphi_i$$

in the sense that

$$\lim_{n \to \infty} \| x - \sum_{i=1}^{n} \xi_i \varphi_i \| = 0 \,.$$

If X and Y are spaces with countable bases (φ_i) and (ψ_i) respectively and A is a bounded linear transformation from X into Y, then A can be represented by an infinite matrix (a_{ij}), with

$$A\varphi_j = \sum_{i=1}^{\infty} a_{ij} \psi_i$$

[2]. In what follows, the basis used for ℓ^p will be given by $\varphi_j = (0, 0, \cdots, 0, 1, 0, 0, \cdots)$ where there is a 1 in the jth place and 0 elsewhere. Similarly for ψ_i. The matrix representations of operators will all be with respect to these bases.

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Proof of the theorem. The details of the proof are given below only for $X = \ell^p, 1 \leq p < \infty$, and $Y = \ell^q, 1 \leq p \leq q < \infty$. The proof for the remaining pairs is similar and is indicated in a remark at the end.

DEFINITION. Let E be the function on $[\ell^p, \ell^q], 1 \leq p \leq q < \infty$, which sends an operator whose matrix is (a_{ij}) into the operator whose matrix is $(a_{ij}\delta_{ij})$, i.e. the non-diagonal matrix elements are replaced by zero and the diagonal elements are unaltered.

LEMMA 1. *E is a projection with $\| E \| = 1$, range the diagonal operators, and null-space the operators with $a_{ii} = 0$, all i.*

Proof. E is additive and homogeneous as easily follows from [2]. $E^2 = E$, and the characterization of the range and null-spaces are apparent.

From the chain

$$\begin{aligned} \infty > \| A \| &= \sup_{\|x\|_p \leq 1} \| Ax \|_q \geq \sup_j \| A\varphi_j \|_q \\ &= \sup_j \| \sum_i a_{ij} \psi_j \|_q \geq \sup_j \| a_{jj}\psi_j \|_q = \sup_j | a_{jj} | \\ &\geq \sup_{\sum |\xi_i|^p \leq 1} (\sum | a_{ii}\xi_i |^p)^{1/p} \geq \sup_{\|x\|_p \leq 1} (\sum | a_{ii}\xi_i |^q)^{1/q} = \| EA \| , \end{aligned}$$

where the last $\geq$ is by Jensen's inequality, we see that E sends bounded operators into bounded operators and, further, $\| E \| = 1$. Also

$$\| EA \| \leq \sup_j | a_{jj} | .$$

In fact,

$$\| EA \| = \sup_j | a_{jj} |$$

because

$$\| EA \| \geq \sup_j \| EA\varphi_j \| = \sup_j | a_{jj} | .$$

LEMMA 2. *The mapping γ from the set of diagonal operators onto ℓ^∞ defined by $\gamma(a_{ii}) = (a_{11}, a_{22}, \cdots)$ is an isometry which carries the compact diagonal operators onto c_0.*

Proof. That γ is an isometry from the diagonal operators onto ℓ^∞ follows from the previous observation that $\| EA \| = \sup_j | a_{jj} |$. Hence it suffices to show that the compact diagonal operators are exactly those with the additional condition $\lim_i | a_{ii} | = 0$. This condition is necessary;

otherwise for some $\varepsilon > 0$ there is an infinite index set I such that $|a_{ii}| \geq \varepsilon$ whenever $i \in I$. Then the bounded sequence $(\varphi_i)_{i\in I}$ would be carried into the sequence $(a_{ii}\psi_i)_{i\in I}$, which has no convergent subsequence, showing (a_{ii}) is not compact. The condition is sufficient because, if $\|x\|_p \leq 1$ then

$$\Big(\sum_{i=1}^{\infty} |a_{ii}\xi_i|^q\Big)^{1/q} \leq (\sup_{i\geq n} |a_{ii}|) \|x\|_q \leq \sup_{i\geq n} |a_{ii}|$$

and [2; Th. 2] applies. The last inequality follows from Jensen's inequality and our assumptions $p \leq q$, $\|x\|_p \leq 1$.

LEMMA 3. *Suppose X is a Banach space with a closed subspace $\mathfrak{M}$ onto which there is a bounded projection E. Let $\mathfrak{N}$ be the null-space of E. Let $\mathfrak{P}$ be any closed linear manifold of X such that if $f \in \mathfrak{P}$ then $f = g + h$, with $g \in \mathfrak{P} \cap \mathfrak{M}$ and $h \in \mathfrak{P} \cap \mathfrak{N}$. Then, given any bounded projection F onto $\mathfrak{P}$, EF is a bounded projection onto $\mathfrak{P} \cap \mathfrak{M}$ such that $\|EF\| \leq \|E\| \|F\|$.*

The proof is an obvious modification of [3; Lemma 1.2.1].

Let $\mathfrak{P}$ be the set of compact operators, $\mathfrak{M}$ the set of diagonal operators, E the projection of Lemma 1, and $\mathfrak{N}$ its null-space. In order to apply Lemma 3 it remains to show: given any compact operator f, Ef and $f - Ef$ are compact. Ef is compact because, if f is compact,

$$\lim_n \Big\|\sum_{i=n}^{\infty} a_{ij}\psi_i\Big\| = \lim_n \Big(\sum_{i=n}^{\infty} |a_{ij}|^q\Big)^{1/q} = 0$$

uniformly in j. This implies $\lim_n |a_{nn}| = 0$, which shows that Ef is compact. Hence $f - Ef$ is compact.

To prove the theorem for $[\ell^p, \ell^q]$, $1 \leq p \leq q < \infty$, assume there is a bounded projection F from $[\ell^p, \ell^q]$ onto $\mathfrak{P}$. By Lemma 3, the restriction of EF to $\mathfrak{M}$ is a bounded projection from $\mathfrak{M}$ onto $\mathfrak{M} \cap \mathfrak{P}$. By Lemma 2 there must be a corresponding bounded projection from ℓ^∞ onto c_0. This contradicts [4; Cor. 7.5]. For the remaining X and Y pairs of the main theorem, the proof is similar except that the existence of expressions for $\|A\|$ in terms of the matrix coefficients (e.g., see [5]) makes some of the work simpler.

Next we extend the theorem to $[U, V]$. Let $\tilde{E}$ be the function on $[U, V]$ defined by $\tilde{E}f = P_2fP_1$ for all f in $[U, V]$. $\tilde{E}$ is linear and homogeneous and bounded. $\tilde{E}^2f = P_2(P_2fP_1)P_1 = P_2fP_1 = \tilde{E}f$ so $\tilde{E}$ is a projection. The range of $\tilde{E}$ is the set of operators g such that $P_2gP_1 = g$ and is isomorphic with $[X, Y]$. The null-space of $\tilde{E}$ is the set of operators h such that $P_2hP_1 = 0$. If Q_i is the projection $I - P_i$, the decomposition $f = g + h$ is given by

$$f = (P_2 + Q_2)f(P_1 + Q_1) = \underbrace{P_2 f P_1}_{g} + \underbrace{(P_2 f Q_1 + Q_2 f P_1 + Q_2 f Q_1)}_{h} .$$

If f is compact, so are g and h. We apply Lemma 3 with $X = [U, V]$, $\mathfrak{M}$ the range of $\tilde{E}$, $\tilde{E}$ acting as the projection E of that lemma, and $\mathfrak{P}$ the set of compact operators from U to V. The conclusion is that if there were a bounded projection F from X to $\mathfrak{P}$, the restriction of $\tilde{E}F$ to $\mathfrak{M}$ would be a bounded projection from $\mathfrak{M}$ onto $\mathfrak{P} \cap \mathfrak{M}$, contradicting our result for $[X, Y]$.

REMARK. The problem of finding a bounded projection onto the compact operators is trivial when all the bounded operators are compact. This happens, for example, for $[\ell^p, \ell^q]$, $\infty > p > q \geq 1$, [2, p. 700], or $p = \infty$, $q = 1$, and for $[c_0, \ell^q]$, $[c, \ell^q]$, $\infty > q \geq 1$. Whether there exists a pair of normed spaces with a bounded proper projection from the bounded operators onto the compact operators seems to be unknown.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA