Journal für die reine und angewandte Mathematik
Herausgegeben von **Helmut Hasse** und **Hans Rohrbach**
Sonderdruck aus Band 248, Seite 117 bis 122
Verlag Walter de Gruyter · Berlin · New York 1971

Partially bounded sets of infinite width[1])

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1. Introduction

Clark [2] introduces these notions for a convex subset A of a real Banach space X: (1) A has *finite width* if A lies between two closed parallel hyperplanes, i. e. if there is a continuous linear functional x^* on X and two real numbers α and β with $\alpha \leqq x^*(x) \leqq \beta$ for all x in A, (2) A has finite width in the direction $x \neq 0$ if there is a constant w_x such that every line parallel to x intersects A in an interval of length no more than w_x, i. e. if $\alpha x + y$ and $\beta x + y$ belong to A then $|\beta - \alpha| \leqq w_x$, and (3) A is *partially bounded* (by K) if there is a finite least upper bound K for the radii of spheres contained in A. In what follows, we assume A is closed.

Clark shows that these conditions are equivalent when X is finite dimensional and asks whether this is true in an infinite dimensional Hilbert space. (Ironically, the closure of the set W on page 615 of [2] furnishes an example of a closed convex body in separable Hilbert space which is partially bounded by 1 but which is of finite width in no direction.)

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are easy to establish. We show here that under various hypotheses, satisfied by many and perhaps by all infinite dimensional Banach spaces, none of the other possible implications hold. Specifically (1) there are partially bounded closed convex bodies of infinite width in any Banach space which has a separable infinite dimensional quotient, and (2) in any Banach space which has an infinite dimensional quotient having a Schauder basis, there is a closed convex set of infinite width but of finite width in some direction.

We note that Clark's discussion of the finite dimensional case is limited to convex bodies, i. e. convex sets with non-void interior. This is reasonable because a convex set with void interior is contained in a proper hyperplane, in the finite dimensional case. This shows that it trivially has all the properties of interest. It seemed to us possible but unlikely that Clark's interest was limited to convex bodies in the infinite dimensional case. However, we do give convex body examples whenever possible.

2. Partially Bounded Sets of Infinite Width

Theorem 1. *Let X be an infinite dimensional Banach space which has a separable infinite dimensional quotient. Then X contains a closed convex set of infinite width which is partially bounded by* 0.

[1]) This research was supported by the Air Force Office of Scientific Research under Grants AFOSR-AF-111-63 and 70-1870 A and by the National Science Foundation under Grant GP-13288.

Proof. First assume that X is separable with $\{x_n\}$ a linearly independent set whose span is dense in X. We apply a generalization of the Gram-Schmidt process: Let x_1^* be a continuous linear functional with $x_1^*(x_1) = 1$. Set $z_1 = \frac{x_1}{\|x_1\|}$ and $z_1^* = \|x_1\| x_1^*$. Let $y_2 = x_2 - z_1^*(x_2)z_1$ and let y_2^* be a continuous linear functional with $y_2^*(z_1) = 0$ and $y_2^*(y_2) = 1$. Set $z_2 = \frac{y_2}{\|y_2\|}$ and $z_2^* = \|y_2\| y_2^*$. Continuing we obtain sequences $\{z_n\}$ and $\{z_n^*\}$ with

$$z_n^*(z_m) = \delta_{mn}, \tag{1}$$

$$\operatorname{span}(z_1, \ldots, z_n) = \operatorname{span}(x_1, \ldots, x_n). \tag{2}$$

That such a biorthogonal sequence could be found in each separable Banach space was known to Banach ([1], page 238). We note in passing that [5], page 214 claims it is an open question as to whether each separable Banach space contains a set with dense span but having no proper subset with dense span. The set $\{z_n\}$ above is such a set.

Define the closed convex set $A = \{x \text{ in } X : z_n^*(x) \geqq 0,\ n = 1, 2, \ldots\}$. Suppose x^* is a non-zero continuous linear functional. Since $\{z_n\}$ has dense span, $x^*(z_m) \neq 0$ for some index m and, because $a z_m$ is in A for each positive number a, $x^*(A)$ is unbounded. Thus A has infinite width.

To see that A is partially bounded by zero, assume that for some $r > 0$, $\{x : \|x - x_0\| < r\} \subseteq A$. There is a vector $z_0 = \sum_1^p a_n z_n$ with $\|x_0 - z_0\| < \frac{r}{2}$; thus $\left\{x : \|x - z_0\| < \frac{r}{2}\right\} \subseteq A$. But, this is false as $\left\|\left(-\frac{r}{4} z_{p+1} + z_0\right) - z_0\right\| < \frac{r}{2}$ yet $z_{p+1}^*\left(-\frac{r}{4} z_{p+1} + z_0\right) < 0$. This establishes the theorem when X is separable.

Next, suppose that there is a closed subspace M of X with $\frac{X}{M}$ separable and infinite dimensional. Let $Q : X \to \frac{X}{M}$ be the quotient map. Construct the set A in $\frac{X}{M}$ as described above and let $B = Q^{-1}(A)$. The set B is closed, convex and is partially bounded by 0 (i. e. has void interior) since A has void interior and Q is an open map.

To see that B has infinite width, let $x^* \neq 0$ be a continuous linear functional on X. There are two cases. If x^* is not zero when restricted to the subspace M, then, since $M \subseteq B$, $x^*(B)$ is not bounded. On the other hand, if $x^*(M) = 0$ then x^* is in $M^\perp = \left(\frac{X}{M}\right)^*$, i. e. there is a non-zero f in $\left(\frac{X}{M}\right)^*$ with $f(x + M) = x^*(x)$ for all x. By the construction of A, $f(A)$ is unbounded and thus $x^*(B)$ is unbounded. Hence B is the desired set in X, which completes the proof.

It is an open question [7] whether each infinite dimensional Banach space has a separable infinite dimensional quotient.

In [8], page 599 we asked whether there is a Banach space X with the property that in X^* weak* and norm convergence are the same for sequences. If there is such a space X, then it does not have a separable infinite dimensional quotient. To see this let M be a closed subspace of X with $\frac{X}{M}$ separable and let $Q : X \to \frac{X}{M}$ be the quotient map. Each bounded sequence in $\left(\frac{X}{M}\right)^*$ contains a weak* convergent sequence $\{f_n\}$ ([3], Theorem 1, page 426) and $\{Q^* f_n^*\}$ is norm convergent by hypothesis. Hence Q^* is a compact operator, so Q is also compact and, being onto, must have a finite dimensional range.

Question: Does every infinite dimensional Banach space contain a partially bounded closed convex set of infinite width?

We modify the above example to have A a closed convex body by using the following lemma.

Lemma 2. *Let X be a Banach space with closed unit sphere S, A a closed convex set partially bounded by K, and $A_\varepsilon = cl(A + \varepsilon S)$ for $\varepsilon > 0$. Then A_ε is partially bounded by $K + \varepsilon$.*

Proof. Since $\operatorname{int}(cl(A + \varepsilon S)) = \operatorname{int}(A + \varepsilon S)$ ([9], Theorem 1.16, page 13), it follows that A_ε is partially bounded by $K + \varepsilon$ if $A + \varepsilon S$ is partially bounded by $K + \varepsilon$.

It is evident that $A + \varepsilon S$ is partially bounded by a constant which is at least $K + \varepsilon$. We wish to show that it is exactly $K + \varepsilon$. Suppose instead that it is partially bounded by $K + \varepsilon + \delta$, $0 < \delta \leqq \infty$. Then there is a sphere B in $A + \varepsilon S$ of radius $K + \varepsilon + \delta_0$, $0 < \delta_0 < \delta$. The center p of B lies in A, for if, instead, it were in $A + \varepsilon S$ but not in A, there would be a continuous linear functional x^* of norm one such that $x^*(p) \geqq x^*(A)$ ([3], Theorem V. 2. 12, page 418). Then

$$\sup\{x^*(x) : x \text{ in } B\} = x^*(p) + K + \varepsilon + \delta_0,$$

since $\sup\{x^*(x) : x \text{ in } B - p\}$ is the radius of $B - p$. Then

$$x^*(p) + K + \varepsilon + \delta_0 = \sup\{x^*(x) : x \text{ in } B\} \leqq \varepsilon + \sup\{x^*(x) : x \text{ in } A\} \leqq x^*(p) + \varepsilon,$$

a contradiction.

Since p is in A, the sphere centered at p of radius $K + \frac{\delta_0}{2}$ contains a point q not in A. Choose a continuous linear functional y^* of norm one such that $y^*(q) \geqq y^*(A)$. Then by an argument like the preceding, any sphere centered at q with radius $\varepsilon + \frac{\delta_0}{2}$ cannot be contained in $A + \varepsilon S$. Thus there are points within

$$K + \frac{\delta_0}{2} + \varepsilon + \frac{\delta_0}{2} = K + \varepsilon + \delta_0$$

of p which are not in $A + \varepsilon S$. This contradiction completes the proof.

Lemma 2 and Theorem 1 yield:

Theorem 3. *Let X be an infinite dimensional Banach space which has a separable infinite dimensional quotient, and let $K \geqq 0$ be any non-negative real number. Then X contains a closed convex body of infinite width which is partially bounded by K.*

3. Finite Width and Finite Width in Some Direction

The next Theorem, although simple, is crucial for our characterization of the properties of finite width and finite width in some direction.

Theorem 4. *Let X be a Banach space and A a closed convex subset of X. Then*

(a) *A is of finite width if and only if $cl(A - A)$ is not all of X,*

(b) *A is of finite width in the direction x if and only if for some scalar α, αx is not in $A - A$, and*

(c) *if A is a closed convex body, then it is of finite width in some direction if and only if it is of finite width.*

Proof. (a) If x is not in $cl(A - A)$, then by the separation theorem ([3], Theorem V. 2. 12, page 418) there is a non-zero continuous linear functional x^* with $x^*(x) \geqq x^*(z)$ for each z in $A - A$. From the symmetry of $A - A$, $x^*(A)$ is bounded so A is of finite width. Conversely if A is of finite width, then there is a non-zero continuous linear functional x^* with $|x^*(a)| \leqq K$ for all a in A and so any x with $|x^*(x)| > K$ is not in $cl(A - A)$.

(b) Suppose that A is of infinite width in the direction x. Then for every scalar α there are vectors a_1 and a_2 in A with $\alpha x = a_1 - a_2$. On the other hand suppose that for each scalar α, $\alpha x = a_1 - a_2$ is in $A - A$. Then the line joining $a_1 = a_2 + \alpha x$ to a_2 is in A and is a segment in the direction x of length α. Thus A is of infinite width in the direction x.

(c) As we have noted, it is easy to see that if A is of finite width, say

$$\alpha_0 \leqq x^*(a) \leqq \alpha_1$$

for a in A, then it is of finite width in some direction; in fact if $x^*(x) = 1$, then $\alpha x + y$ and $\beta x + y$ both in A imply that $|\beta - \alpha| \leqq \alpha_1 - \alpha_0$.

Suppose that A is a closed convex body of finite width in the direction x. By (b) there is a scalar α with αx not in $A - A$. Since $A - A$ is a convex body, there is a non-zero continuous linear functional x^* with $x^*(\alpha x) \geqq x^*(A - A)$ ([3], Theorem V. 2. 8, page 417). Since $A - A$ is symmetric, $x^*(-\alpha x) \leqq x^*(A - A)$. Thus $A - A$ is of finite width and so A is of finite width.

Theorem 3 and Theorem 4 (c) yield:

Corollary 5. *Let X be a Banach space with an infinite dimensional separable quotient. Then there is a closed convex body in X which is partially bounded but not of finite width in any direction x.*

4. Clark's Theorem

Using Theorem 4 we now give a comparatively brief and elementary proof of Clark's principal result that (1), (2) and (3) are equivalent properties in E^n and so are equivalent in any finite dimensional normed linear space.

Theorem 6 (Clark). *The properties (1), (2) and (3) are equivalent in E^n.*

Proof. If A has void interior, it is contained in a proper subspace ([4], Theorem 4, page 16) and then (1), (2) and (3) all hold for A. Assuming then that A is a body, (1) and (2) are equivalent by Theorem 4. Since it is easy to see that (1) implies (3) it remains to show that (3) implies (2). We show that not-(2) implies not-(3).

Suppose A is a convex subset of infinite width in every direction and let $M > 0$ be otherwise arbitrary. Since A has infinite width in each direction we may choose a segment in A of length M. Choose a basis $\{e_1, \ldots, e_n\}$ for E^n so that the segment is $\frac{M}{2}[-e_1, e_1]$, where $[x, y]$ denotes the line joining x and y. There is a segment in A of length M parallel to e_2 and so of the form $x_1 + \frac{M}{2}[-e_2, e_2]$. Since A is convex it contains the segment $\frac{x_1}{2} + \frac{M}{2^2}[-e_1, e_1]$, obtained from averaging x_1 and $\frac{M}{2}[-e_1, e_1]$, and the segment $\frac{x_1}{2} + \frac{M}{2^2}[-e_2, e_2]$, obtained from averaging 0 and $x_1 + \frac{M}{2}[-e_2, e_2]$. By translation we may choose a new basis and so suppose that $x_1 = 0$.

Proceeding by induction, suppose for $1 \leq k < n$, A contains the k-dimensional "cross" $\bigcup_{j=1}^{k} \frac{M}{2^k}[-e_j, e_j]$. There is a segment $x_k + \frac{M}{2^k}[-e_{k+1}, e_{k+1}]$ in A and so A contains the $(k+1)$-dimensional "cross" $\frac{x_k}{2} + \bigcup_{j=1}^{k+1} \frac{M}{2^{k+1}}[-e_j, e_j]$ and by a change of basis we may suppose that $x_k = 0$. Finally, A contains the convex hull of $\bigcup_{j=1}^{n} \frac{M}{2^n}[-e_j, e_j]$ relative to a suitable basis. Thus, since M is arbitrary, A contains arbitrarily large replicas of the unit sphere in l_1^n, hence it contains arbitrarily large E^n spheres and is not partially bounded.

5. Finite Width in Some Direction Does Not Imply Finite Width

We finally show, under hypotheses perhaps stronger than Theorem 1, that properties (1) and (2) are not equivalent in general, in contrast to their equivalence for convex bodies as shown in Theorem 4.

Theorem 7. *Let X be a Banach space having an infinite dimensional quotient space with a Schauder basis. Then there is a closed convex set in X which is of infinite width but has finite width in some direction.*

Proof. It is easy to see that if A is a set in a quotient $\frac{X}{M}$ of X which has property (2) but not property (1), then $Q^{-1}(A)$ is such a set in X, Q being the quotient map of X onto $\frac{X}{M}$. So it suffices to construct such a counter-example in any infinite dimensional Banach space having a Schauder basis.

Let X be infinite dimensional with Schauder basis $\{x_n\}$, $\| x_n \| = 1$ for all n. First suppose that there is an element $x = \Sigma\, \alpha_n x_n$ with $\Sigma\, | \alpha_n |$ divergent. There is a permutation σ of the integers with $\Sigma\, | \alpha_{\sigma(n)} - \alpha_{\sigma(n+1)} |$ divergent. Define the continuous linear functionals x_m^* by $x_m^*(\Sigma\, \beta_n x_n) = \beta_{\sigma(m)} - \beta_{\sigma(m+1)}$ and let $A = \{y \text{ in } X : x_m^*(y) \geqq 0 \text{ for all } m\}$. The set A is a closed, convex cone. The vectors $x_{\sigma(n+1)} = \sum_{j=1}^{n+1} x_{\sigma(j)} - \sum_{j=1}^{n} x_{\sigma(j)}$ are in $A - A$ and $A - A$ is then dense. Thus A is of infinite width by Theorem 4.

We claim that $x = \Sigma\, \alpha_n x_n$ is not in $A - A$ and therefore, by Theorem 4, that A has finite width in the direction x. If instead $x = a_1 - a_2$, with a_1 and a_2 in A, then $| x_m^*(x) | \leqq x_m^*(a_1) + x_m^*(a_2) = x_m^*(z)$, with $z = a_1 + a_2$ in A. Let $z = \Sigma\, z_i x_i$. From

$$z_{\sigma(m)} - z_{\sigma(m+1)} \geqq | \alpha_{\sigma(m)} - \alpha_{\sigma(m+1)} |$$

we have

$$z_{\sigma(1)} \geqq z_{\sigma(2)} + | \alpha_{\sigma(1)} - \alpha_{\sigma(2)} | \geqq z_{\sigma(3)} + | \alpha_{\sigma(1)} - \alpha_{\sigma(2)} | + | \alpha_{\sigma(2)} - \alpha_{\sigma(3)} | \geqq \cdots$$
$$\geqq z_{\sigma(m+1)} + \sum_{j=1}^{m} | \alpha_{\sigma(j)} - \alpha_{\sigma(j+1)} |$$

which is a contradiction since the right hand side tends to infinity with m.

There remains the case where $\Sigma\, | \beta_n |$ converges for each $\Sigma\, \beta_n x_n$ in X. Then the map which takes each $\{\beta_n\}$ in l_1 to $\Sigma\, \beta_n x_n$ in X is a one-to-one continuous map of l_1 onto X and X is isomorphic to l_1. It will thus be enough to establish the result for l_1. There is a continuous linear map of l_1 onto, say, l_2 ([6], page 63) and as an appropriate counter-example has been constructed in l_2 above, this generates the desired example in l_1 as mentioned in the beginning of the proof. This completes the proof.

Let S be a compact Hausdorff space containing infinitely many points and consider the Banach space $C(S)$. It is interesting to construct directly natural counter-examples in $C(S)$ showing that (3) $\not\Rightarrow$ (2) and that (2) $\not\Rightarrow$ (1).

First, let s_0 be a point of S which is not isolated and let $I = \{f \text{ in } C(S) : f(s_0) = 0\}$. Then I is a quotient of $C(S)$ and the set $I_0 = \{f \text{ in } I : f \geqq 0\}$ is a closed convex set in I having void interior, and so partially bounded. The set I_0 is of infinite width in each direction in I. To see this let $g \neq 0$ in I be given. Then for each $M > 0$, with $f = M\,|\,g\,|$ we have $\alpha g + f$ in I_0 for $|\,\alpha\,| \leqq M$. Thus $I_0 \oplus sp(1)$ is a set in $C(S)$ with property (3) but not (2).

Second, let $\{s_{\alpha_i}\}$ be a countable set of distinct points in S which form a convergent net in S. Let

$$A = \{f \text{ in } C(S) : f \geqq 0 \text{ and } f(s_{\alpha_i}) - f(s_{\alpha_{i+1}}) \geqq 0 \text{ for all } i\}.$$

We see that $A - A$ is a subalgebra of $C(S)$ which separates points and contains the constants and thus, by the Stone-Weierstrass theorem, is dense in $C(S)$. So A is of infinite width by Theorem 4. Let U_i be a compact neighborhood of s_{α_i} with $\{U_i\}$ mutually disjoint. There are continuous functions f_i on S with (a) $f_i(s) = 0$ for s not in U_i, (b) $0 \leqq f_i \leqq 1$, and (c) $f_i(s_{\alpha_i}) = 1$. Then for scalars $\{z_i\}$ with $\lim z_i = 0$ and $\Sigma\,|\,z_{i+1} - z_i\,|$ divergent, $g = \Sigma\, z_i f_i$ is in $C(S)$ and one can show, exactly as in Theorem 7, that g does not belong to $A - A$. Hence A is of finite width in the direction g.

Question. Does every infinite dimensional Banach space contain a closed convex set which is:

(a) partially bounded but having finite width in no direction, or

(b) of finite width in some direction but of infinite width?

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Eingegangen 10. November 1969