OPTIMAL GAMBLING SYSTEMS FOR FAVORABLE GAMES

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INTRODUCTION

In the last decade it was found that the player may have the advantage in some games of chance. We shall see that blackjack, the side bet in Nevada-style Baccarat, roulette, and the wheel of fortune all may offer the player positive expectation. The stock market has many of the features of these games of chance [5]. It offers special situations with expected returns ranging above an annual rate of 25% [23].

Once the particular theory of a game has been used to identify favorable situations, we have the problem of how best to apportion our resources. Paralleling the discoveries of favorable situations in particular games, the outlines of a general mathematical theory for exploiting these opportunities has developed [2, 3, 10, 13].

We first describe the favorable games mentioned above, those being the ones with which the author is most familiar. Then we discuss the general mathematical theory, as it has developed thus far, and its application to these games. Detailed knowledge of particular games is not needed to follow the exposition. Each discussion of a favorable game in Part I motivates a concluding probabilistic summary of that game. These summaries suffice for the discussion in Part II so that a reader who has no interest in a particular game may skip directly to the summary.

References are provided for those who wish to explore particular games in detail. For the present, a favorable game means one in which there is a strategy such that $P(\lim S_n = \infty) > 0$ where S_n is the player's capital after n trials.

PART I. FAVORABLE GAMES

1. BLACKJACK

Blackjack, or twenty-one, is a card game played throughout the world. The casinos in Nevada currently realize an annual net profit of roughly eighty million dollars from the game. Taking a price/earnings ratio of 15 as typical for present day common stocks, the Nevada blackjack operation might be compared to a \$ 1.2 billion corporation.

To begin the game a dealer randomly shuffles n decks of cards and players place their bets. (The value of n does not materially affect our discussion. It generally is 1, 2, or 4, and we shall use 1 throughout.) There are a maximum and a minimum allowed bet.

The minimum insures a positive probability of eventual ruin for the player who continues to bet. The maximum protects the casino from large adverse fluctuations and in particular prevents the game from being beaten by a martingale (e.g. doubling up), especially one starting with a massive bet. In fact, without a maximum, a casino

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[1] The research for this paper was supported in part by the Air Force Office of Scientific Research through Grant AF-AFOSR 1113-66.
The paper is intended in large part to be an exposition for the general mathematical reader with some probability background, rather than for the expert.

with finite resources can in general be ruined a.s. (almost surely) by a player with infinite resources. The player simply bets enough at each trial so that the casino is ruined if it loses that trial. A practical way for the player to have infinite resources would be for the casino to extend unlimited credit for the finite time it might be needed.

The players' hands are dealt after they have placed their bets. Each player then uses skill in his choice of a strategy for improving his hand. Finally, the dealer plays out his hand according to a fixed strategy which does not allow skill, and bets are settled. In the case where play begins from one complete randomly shuffled deck, an approximate best strategy (i.e. one giving greatest expected return) was first given in 1956 [1].

Though the rules of blackjack vary slightly, the player following [1] typically has the tiny edge of $+ .10\%$. (The pessimistic figure of $- .62\%$ cited in [1] was erroneous and may have discouraged the authors from further analysis.) These mathematical results were in sharp contrast to the earlier and very different intuitive strategies generally recommended by card experts, and the associated player disadvantage of two or three per cent. We call the best strategy against a complete deck the *basic strategy*. Determined in 1965, it is almost identical with the strategy in [1] and it gives the player an edge of $+ 0.13\%$ [22].

If the game were always dealt from a complete shuffled deck, we would have repeated independent trials. But for compelling practical reasons, the deck is not generally reshuffled after each round of play. Thus as successive rounds are played from a given deck, we have sampling without replacement and dependent trials. It is necessary to show the players most or all of the cards used on a given round of play before they place their bets for the next round. They can then use this knowledge of which cards have been played both to sharpen their strategy, and to more precisely estimate their edge. (The strategies for various card counting procedures, and their expectations, were determined directly from probability theory with the aid of computers. The results were reverified by independent Monte Carlo calculations.)

For a given card counting procedure and associated strategy, there is a probability distribution F_c describing the player's expectation on the next hand, provided c cards have been counted. As c increases, F_c spreads out. (This is a theorem, whose proof resembles that for the similar theorem in Baccarat, mentioned in [24], page 316). This spread in F_c can be exploited by placing large bets when the expectation is positive and small bets when it is negative. Part II indicates how best to do this.

If the basic strategy is always used, $E(F_c) = + 0.13\%$, just as from a complete deck. But if an improved strategy, based on the card count, is used, $E(F_c)$ increases as c increases, approaching values of one to two per cent or more.

Ties, in which no money is won or lost, may be discounted. They occur about one tenth of the time. Most, but not all, of the other outcomes result in the player either winning or losing an amount equal to his original bet.

The conditional means $E(F_c \mid F_{c-k})$; $k = 1, 2, \ldots, c$, of the successive F_c are non-decreasing. The F_c are dependent; in particular when a deck "goes good", it tends to stay good.

Probabilistic summary

To a good first approximation, Blackjack is a coin toss where the probability p of success is selected independently on each trial from a known distribution F (which is a suitably weighted average of the F_c) and announced before each trial.

A more accurate model considers that the p's are dependent in short consecutive groups, corresponding to successive rounds of play from the same deck. Another more accurate observation is that insurance, naturals, doubling down, and pair splitting, each win or lose an amount different from the amount initially bet. We do not consider this more accurate model in part II because the improvement in results is slight and the increase in complexity is considerable.

2. BACCARAT

The terms Baccarat and Chemin de Fer are used, sometimes interchangeably, to refer to several closely related variants of what is essentially one card game. The game is currently popular in England and France, where it is sometimes played for unlimited stakes. It is also played in Nevada. The game-theoretic aspects of Baccarat have been discussed in [11, 14]. The Nevada game is analyzed in [24] which includes results of extensive computer calculations.

The studies of Baccarat show that the available bets generally offer an expectation on the order of -1%. The use of mixed strategies, to the very limited extent that this is possible in some variants of the game, has but slight effect on the expectation. Despite the resemblances between Baccarat and Blackjack, the favorable situations detected by perfect card counting methods are not sufficient to make the game favorable. Thus Baccarat is not in general a favorable game.

The game as played in Nevada sometimes permits certain side bets. The minimum on the side bets was observed to be \$ 5 to \$ 20 and the maximum was \$ 200. The bets either won nine times the amount bet or lost the amount bet. The game was played with eight well shuffled decks dealt from a dealing box, or shoe. Using the card counting techniques described in [24], the side bets were favorable about 20% of the time. When they were favorable, the expectations ranged as high as $+100\%$. The expectation initially was about -5% and as the number c of cards seen increased, the distribution F_c of expectations spread out ([24], page 316) as in Blackjack. In practice the betting methods discussed in part II, in which the bet increased with the expectation, doubled initial capital in twenty hours.

Unlike the Blackjack player, the Baccarat side bettor has no strategic decisions to make so $E(F_c)$ does not vary as c changes. When the expectation of the side bet falls below a certain value, it is best to make a "waiting" bet on one of the main bets. There are either two or four side bets, similar and dependent. How to apportion funds on the side bets is complicated by the fact that there are several of them. These complexities are treated in [24].

Probabilistic summary

When only one side bet is available, the pay-off for a one unit side bet is either $+9$ or -1. If p is the probability of success, we may suppose that p is selected independently from a known distribution F and announced before each trial. When several side bets are available, the situation is more complex. It illustrates the general setting of [3], page 65.

As in Blackjack, a more accurate model considers that the p's are dependent in consecutive groups, corresponding to successive rounds of play from the same ensemble of (eight) decks. It also considers the effect of waiting bets.

The situation here is more complex than in Blackjack. First, it is important to exploit any opportunities of making simultaneous bets on two or more favorable side bet situations. Second, the pay-off is never one to one.

3. ROULETTE

Roulette has long been the prototype of unbeatable gambling games. It is normally regarded as a repeated independent trials process which generates at each trial precisely one from a set of random numbers. In Monte Carlo these numbers are 0, 1, 2, . . . , 36. Players may wager on particular conventional subsets of random numbers (e.g. the first dozen, even, {27}, etc.), winning if the number which comes up is a number of the chosen subset. A player may wager on several subsets simultaneously, and each bet is settled without reference to the others. The expectation of each bet is negative (in Nevada generally — 5.26%, except for one worse bet, and in Monte Carlo — 1.35%.) Thus it has been long known that the classical laws of large numbers insure that the player will with probability one fall behind and stay behind, tending to lose in the long run at a rate close to the expectation of his bets.

Despite this, Henri Poincaré and Karl Pearson each examined roulette. Poincaré ([20], pages 69–70, pages 76–77; [21], pages 201–203; [9], pages 61–62) supposes that the uncertainty in initial conditions (e.g. the angular position and velocity of the ball and of the rotor at a given time) leads to a continuous probability density f in the ball's final position. He shows by an argument involving continuity only that if f has sufficient spread, then the finitely many final ball positions are to very high approximation equally likely.

Karl Pearson statistically analyzed certain published roulette data and found very significant patterns. In particular Pearson says, "If Monte Carlo roulette had gone on since the beginning of geological time on this earth, we should not have expected such an occurrence as this fortnight's play to have occurred *once* on the supposition that the game is one of chance." And again, "To sum up, then: Monte Carlo roulette . . . is. . . the most prodigious miracle of the nineteenth century." I've been told that it was later learned that the roulette data was supplied for a newspaper by journalists hired to sit at the wheel and record outcomes. The journalists instead simply made up numbers and submitted them. It was their personal bias that Pearson detected as statistically significant.

It brings to mind David Hume's essay *Of Miracles*: "No testimony is sufficient to establish a miracle, unless the testimony be of such a kind that its falsehood would be more miraculous than the fact that it endeavors to establish. . . . it is nothing strange . . . that men should lie in all ages."

Poincaré assumed a mechanically perfect roulette wheel. However, wheels sometimes have considerable bias due to mechanical imperfections. Some observed instances and their exploitation are discussed in detail in [25].

In Blackjack and Baccarat, we used the following fundamental principle: The payoff random variables, hence the favorability of a game to an optimal player, depend on the information set used to determine the optimal strategy. For instance, if used cards are ignored in Blackjack, then we simply have Bernoulli trials with $p = +0.13\%$. However, as more card counting information is employed, the distribution of p spreads out (has more structure), its expected value increases, and it can be more effectively exploited. The roulette system we now describe illustrates the use of an enlarged information set.

Play at roulette begins when the croupier launches the ball on a circular track which inclines towards the center so the ball will fall into the center when it slows down sufficiently. The center contains a rotor with a circle of congruent numbered pockets rotating in the opposite direction to the ball. The ball eventually slows and falls from its track on the stator, spiralling into the moving rotor and eventually coming to rest in a numbered pocket, the "winning number". Bets may generally be placed until the ball leaves its track. This is crucial for what follows.

A collaborator and I tried to use the mechanical perfection of the wheel – the very perfection needed to eliminate the bias method – to gain positive expectation. Our basic idea was to determine an initial position and velocity for the ball and rotor. We then hoped to predict the final position of the ball in much the same way that a planet's later position around the sun is predicted from initial conditions, hence the nickname "the Newtonian method".

The Newtonian method occurred to me in 1957, and by 1961 the work described here had been completed. Although the wheel of fortune device was mentioned in LIFE Magazine, March 27, 1964, pp. 80–91, we pointedly did not mention the roulette work there. However, we do so in [22], page 181–182. The Newtonian method is also mentioned in the significant book by R. A. Epstein, *The Theory of Gambling and Statistical Logic*, Academic Press, pp. 135–136, (1967).

The stator has metal deflectors placed to scatter the ball when it spirals down and the pockets are separated by vertical dividers ("frets") which also introduce scattering. These scatterings were measured and found to be far from sufficient to frustrate the Newtonian approach. However, there were additional sources of randomness which did frustrate this approach. (We never satisfactorily identified these causes and can only speculate – perhaps the causes included minute imperfections in track or ball or high sensitivity of the coefficient of friction to dirt or atmospheric humidity.)

We were led to a variation we called the quantum method. If a roulette wheel is tilted slightly the ball will not fall from a sector of the track on the "high" side. The effect is strong with a tilt of just 0.2°, which creates a forbidden zone of a quarter to a third of the wheel. The non-linear differential equation governing the ball's motion on the track is the equation for a pendulum which at first swings completely around its pivot, but is gradually slowed by air resistance. (It is illuminating to sketch the orbits of the equation, as indicated in [4], page 402, problem 3.) The experimental orbits of angle versus time could be plotted easily in the laboratory by taking a movie of the system in motion, along with a large electric clock whose hand swept out one revolution per second!

The existence of a forbidden zone partially quantizes the angle at which the ball can exit, and hence quantizes the final angular position of the ball on the rotor. The physics involved suggests that the quantization is in fact very sharp: Suppose the ball is going to exit beyond the low point of the tilted wheel. Then it must have been moving faster than a ball exiting at the low point, so it reaches its destination sooner. But it has also gone farther, and the two effects tend to cancel. They in fact cancel very well. A similar argument shows that balls which exit before the low point have been slower, hence later, offsetting the fact they have not gone as far. Observation verifies the conclusions of this heuristic argument.

The sharp quantization of ball final position, as a function of initial conditions, makes remarkably accurate prediction possible.

Using algorithms, it was possible by eye judgements alone to estimate the ball's final position three or four revolutions before exit (perhaps five to seven seconds before exit, which was ample time in which to bet) well enough to have a + 15% expectation on each of the five most favored numbers. A cigarette pack sized transistorized computer which we designed and built was able to predict up to eight revolutions in advance. The expectation in tests was + 44%.

One third of the Nevada roulette wheels which we observed had the desired tilt of at least 0.2°. The input to the computer consisted of four push-button hits: two when the 0 of the rotor crossed a fiducial mark during successive revolutions and two when the ball crossed a fiducial mark on successive revolutions. The decay constants of ball and rotor, approximately constant over the class of wheels observed, had been determined earlier by simple observations.

The ultimate weakness of the system was that the house could foil it by forbidding bets after the ball had been launched.

Probabilistic summary

Roulette on a slightly tilted wheel is repeated independent trials. At each trial the player may wager on one or more subsets of the finitely many elementary outcomes. A wager on a subset wins if and only if it contains the elementary outcome that occurs. There are subsets with expectations of 44%. Our procedure in practice was to bet on one of eight neighborhoods of five numbers. Thus the payoff for a bet of .2 units on each of five numbers was either -1 or $+6.2$. The expectation of $+44\%$ corresponds to a probability of success of .2. We remark that our knowledge of p increases with the sample size.

4. THE WHEEL OF FORTUNE

The wheel of fortune, featured in many Nevada casinos, is a six foot vertical wheel with horizontal equally spaced pegs in its rim. As the wheel spins, a rubber flapper strikes successive pegs, slowing the wheel. There are generally 48 to 54 spaces between the pegs, numbered with 1s, 2s, 5s, 10s, 20s, and two distinct 40s. A player betting a unit on one of these outcomes is paid that number of units if his outcome occurs. The wheel behaves to good approximation as though a constant increment of energy is lost each time a peg passes the flapper. Thus θ, the total angle of rotation, is proportional to the energy E, which equals $I\omega^2$, where I is the moment of inertia and ω is the angular velocity of the wheel.

In practice, a transistor timing device of match box size (a "spinoff" from the roulette technology) produced a faint click a chosen time after a push-button was hit. The button was hit when a specified 40 passed the flapper. The timer was set so the click was approximately when the second 40 reached the flapper. If it clicked after the second 40 reached the flapper, the wheel was "fast" and would go farther than average before stopping. If it clicked before the second 40 reached the flapper, the wheel was "slow".

For a given timer setting, a table was constructed empirically, giving the approximate final position of the wheel as a function of the number of spaces the second 40 was fast or slow when the click was heard.

In practice one could determine with certainty which of the two 40s could not occur. Thus, one could always bet on the "right" 40. On a wheel observed in the Riviera Hotel there were 50 numbers, including 22 ones, 14 twos, 7 fives, 3 tens, 2 twenties and 2 forties. Betting on the "right" 40 would win on average 80 units in 50 trials and lose 48, for an expectation of 32/50 or 64%.

Probabilistic summary

Ignoring obvious refinements, we have repeated independent trials with probability $p = 1/25$ of success at each trial, a payoff for a 1 unit bet of -1 or $+40$, and an expectation of $+64\%$.

5. THE STOCK MARKET

The stock market is a natural economic object for mathematical analysis because vast quantities of precise historical data are available in numerical form. There have been many attempts to mathematically predict future price behavior, using as a basis various subsets of the available information. Most notable are the attempts to predict future prices from past price behavior. These attempts have caused the view to be widespread in academic circles that, to first order, common stock prices are a random walk and changes in common stock prices are log normally distributed with a certain mean and standard deviation [5].

Practitioners hotly contest this view. Part of the dispute is caused by practitioners who are unwilling or unable to test their claims scientifically and part of it is due to the success of a few practitioners who use much more information than past price history alone. A recent study suggests strongly that "relative strength" in a price series is continued and, consequently, that past prices do have some value in predicting future prices [16].

Whether or not we can predict the future course of stock prices[1], there are investments in combinations of securities which can yield high expected return [23]. These investments involve convertible securities. A convertible security is one which, in some cases with the addition of money, is exchangeable (per share) for a certain number of shares of another security. Convertible securities include convertible bonds, convertible preferreds, stock options, stock rights, and warrants. There are several billion dollars worth of convertibles listed on the New York and American Stock Exchanges.

The analysis of other convertibles follows from the analysis of the common stock purchase warrant. We therefore restrict ourselves to these in our discussion, and shall refer to them simply as warrants.

A warrant is the right or option to buy a certain number of shares of common for a certain price, until a certain expiration date (warrants which do not expire are called perpetual). The terms ordinarily read: A warrants plus E dollars buy C shares until D date. To avoid normalization problems, we suppose $A = C = 1$. Then E is the "exercise price" of the warrant. The prices of warrant and common are related and it is this which allows successful investments. One observes: (1) The price W of the warrant should increase as the price S of the stock increases. (2) If $W + E < S$, warrants can be bought and common sold short, simultaneously. The warrants are then converted to common which is delivered against the short position. Neglecting commissions, a profit of $S - W - E$ per warrant results. The purchase of warrants tends to increase W and the sale of common tends to decrease S, until $W + E \geq S$. Thus $W \geq S - E$ normally holds. (3) The common has advantages over the warrant such as possible dividends, or voting rights, hence we also normally expect $W < S$.

Thus for practical purposes points (S, W) representing (nearly) simultaneous prices of a common stock and its warrant are confined to the part of the positive quadrant between the lines $W = S$ and $W = S - E$.

The prices W and S at a future time are random variables but they are related. As $E(S)$ increases we would expect, and past history verifies ([12, 23]), that $E(W)$ tends to increase. In fact the points (S, W) tend to lie on certain curves which depend

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[1] The great mathematician Karl Friedrich Gauss was successful in the market but we have little knowledge of his methods. On a basic salary of 1000 thalers per year he left an estate in cash plus securities of 170,857 thalers ([7], page 237).

on several variables, most notably the time remaining until expiration of the warrant. Thus, although we may not know the price S of the common, or the price W of the warrant at a given future time, we do know that (W, S) is near one of these curves. The family of curves qualitatively resembles the family $W = (S^z + E^z)^{1/z} - E$, where $z = 1.3 + 5.3/T$ and T is the number of months remaining until expiration.

A historical study of expiring warrants (from here on we limit ourselves for convenience to warrants traded on the American Stock Exchange) suggests that during the last two years or so before expiration they tend to trade at prices which are much too high. For instance the average loss from buying each of a certain 11 listed warrants 18 months before expiration and holding until 2 months before expiration was 46.0%, an annual rate of 34.5% ([23], page 37). Thus selling warrants short seems to yield high expectation. However, it also happens to result in occasional large losses which, by the criterion of Part II, are extremely undesirable despite the high overall expectation. We can sharply reduce this high variance and yet retain a high expectation by using the so-called warrant hedge. The technique is to simultaneously sell short overpriced warrants and buy common in a fixed ratio (generally from one to three warrants will be shorted for each share of common bought). The position is held until just before expiration of the warrant (at which time the warrant sells at a "correct" price) and then it is liquidated.

Here is the rationale. We are mixing two investments with positive annual expectations of say 34.5% for the warrants and 10% for the common, resulting in an investment whose overall expectation must therefore be somewhere between these figures. (We suggest 10% for the common because this approximates the observed mean rate of return from common stocks during this century due to price appreciation plus dividends.) Buying the common leads to a gain when the common rises and a loss when it falls whereas shorting warrants leads to a gain when the common falls and leads to a loss only if the common rises substantially. Thus the risks tend to cancel out. In fact, the hedge generally yields a profit upon expiration of the warrant, for a wide range of prices of the common.

If we make assumptions about the probability distribution of the price of the common at expiration of the warrant, we get more precise information about the random variable representing the payoff from the hedge. Let the probability measure P with support $[0, \infty)$ describe the distribution of the stock price S_f at expiration. Then

$$E(S_f^n) = \int_0^\infty x^n \, dP(x).$$

Let S_0 be the present price and let E be the exercise price. Assume that $P(S_f \geqq S_0 + t) \geqq P(S_f \leqq S_0 - t)$ for each $t \geqq 0$, i.e. for any t, the chance of a price rise of at least t is no less than the chance of a price drop of at least t. This is a very weak assumption. Note that it does imply $E(S_f) \geqq S_0$.

Just before expiration $W_f \doteq 0$ if $S_f \leqq E$ and $W_f \doteq S_f - E$ if $S_f > E$. Thus the final gain from shorting a warrant at W_0 is W_0 if $S_f \leqq E$ and is $W_0 - S_f + E$ if $S_f > E$. The gain from buying a share of common at S_0 is, of course, $S_f - S_0$.

Hence if we assume one share of common is purchased at $.5E$ and one warrant is shorted at $.2E$, the final gain G_f is $S_f - .3E$ if $S_f \leqq E$ and $.7E$ if $S_f > E$. A standard measure-theoretic argument yields $E(G_f) \geqq .2E$. Using 100% margin, the percent profit is $E(G_f) / .7E > 28\%$. With 100% margin on the warrants and 70% margin on the common, it is at least $.2E / .55E > 36\%$. With 70% margin on each, it is at least $.2 / .49 > 40\%$, an annual rate of more than 20% if the warrant expires in two years.

It is interesting to calculate $E(G_f)$ by assuming that S_f is log normally distributed. Letting $s_f = S_f / E$, we thus assume that s_f has the density function

$f(x) = (x\,\sigma\sqrt{2\pi})^{-1} \exp\left[-(\ln x - \mu)^2 / 2\sigma^2\right]$, where μ and σ are parameters depending on the stock. We note $E(s_f) = \exp(\mu + \sigma^2/2)$.

If t is the time in months remaining until expiration (which is when s_f is realized), then we assume $\mu = \log s_0 + mt$ and $\sigma^2 = a^2 t$, where $S_0 = s_0 E$ is the present stock price and m and a are constants depending on the stock. Thus $E(s_f) = s_0 \exp[(m + a^2/2)t]$. A mean increase of 10% per year is approximated by setting $12(m + a^2/2) = .1$. If we estimate a^2 from past price changes we can solve for m.

Letting $w_f = W_f/E$, where W_f is the final warrant price, a calculation yields $E(w_f) = E(s_f)\,N(\mu/\sigma + \sigma) - N(\mu/\sigma)$, where N is the normal distribution. (Compare the equivalent expression from pp. 464–466 of [5].) Now suppose that $s_0 = .5$, that a has the realistic value of $.1 s_0$ or $.05$, and that $12(m + a^2/2) = .1$, whence $m = .085/12$. Then for $t = 24$ we have $\sigma = \sqrt{.06} = .245$ and $\mu = \log .5 + .17 = -.523$. This yields $E(w_f) = .0015$ and $E(s_f) = .61$, whence $E(G_f) = .20 + .11 = .31$. Thus the profit, with 70% margin on both warrant and common, is .31/.49 or 63.3% and the annual rate is 31.6%. Note that the warrant is virtually worthless!

Instead of selling one warrant short and buying one share of common, we can sell short w warrants and buy s shares of common. Neglecting commissions, which we do throughout for simplicity, the gain G at any point (S, W) is $s(S - S_0) - w(W - W_0)$. Thus the line $G = 0$, the zero profit line, is the line through (W_0, S_0) with positive slope $s/w = 1/m$. We call m the *mix*. Points below the zero profit line represent gain and points above it represent loss. If $1 < m < \infty$, the zero profit line intersects the S axis at $S_0 - m W_0$ and it intersects the line $W = S - E$ at $S = [m(W_0 + E) - S_0] / (m - 1)$; $W = (m W_0 + E - S_0) / (m - 1)$.

When the warrant expires the hedge position will yield a profit if S_f is between the S values of the two intersections and it will yield a loss if S_f is beyond the intersections. For instance, if $S_0 = .5E$ and $W_0 = .2E$, the choice $m = 2$ insures a final profit if $.1E < S_f < 1.9E$. Such safety is characteristic of the warrant hedge.

The final gain G_f is $s(S_f - S_0) + wW_0$ if $S_f \leqq E$ and it is $s(S_f - S_0) + w(W_0 + E - S_f)$ if $S_f > E$. Thus as a function of S_f it is an inverted "V" with apex above $S_f = E$. With 100% margin, the initial investment is $s\,S_0 + w\,W_0$ so the gain per unit invested is $g_f = G_f/(s\,S_0 + w\,W_0)$. With margin of α on the common and β on the warrants it is $g_f = G_f/(\alpha\,s\,S_0 + \beta\,w\,W_0)$.

We have assumed so far that a hedge position is held unchanged until expiration, then closed out. This static or "desert island" strategy is not optimal. In practice intermediate decisions in the spirit of dynamic programming lead to considerably superior dynamic strategies. The methods, technical details, and probabilistic summary are more complex so we defer the details for possible subsequent publication.

Probabilistic summary

The warrant hedge may offer high expectation with low risk. The gain per unit g_f is $g_f = [(S_f + S_0) + m\,W_0] / (\alpha\,S_0 + \beta m\,W_0)$ when $S_f \leqq E$ and $g_f = [(S_f - S_0) + m(W_0 + E - S_f)] / (\alpha\,S_0 + \beta m\,W_0)$ if $S_f > E$. The gain per unit depends only on the random variable S_f. This has an unknown distribution but it can be estimated. The other quantities are constants depending on circumstances.

PART II. A MATHEMATICAL THEORY FOR COMMITTING RESOURCES IN FAVORABLE GAMES

I. INTRODUCTION: COIN TOSSING

Suppose we are confronted with an infinitely rich adversary who will match all bets we make on repeated independent trials of a biased coin (whose two outcomes are "heads" and "tails".) Assume that we have finite capital X_0, that we bet B_i on the outcome of the ith trial, where X_i is our capital after the ith trial, and that the probability of heads is p, where $\frac{1}{2} < p < 1$. (This is approximately the situation in Nevada blackjack, except that the game is played with a "mix" of biased coins.) Our problem is to decide how much to bet at each trial. A classic criterion is to choose B_i so that our expected gain $E(X_i - X_{i-1})$ is maximized at each trial, which is equivalent to maximizing $E(X_n)$ for all n.

Define T_j by $T_j = 1$ if the jth trial results in success and $T_j = -1$ if the jth trial results in failure. Then $X_j = X_{j-1} + T_jB_j$, $j = 1, 2, \ldots$, and $X_n = X_o + \sum_{j=1}^{n} T_jB_j$.

We assume that T_j, X_j, and B_j are all random variables on a suitable sample space Ω. If, for example, B_j is a function of $X_0, X_1, \ldots, X_{j-1}$ as it is in the common gambling systems, e.g. Martingale, Labouchere, etc. (note that $B_k = | X_k - X_{k-1} |$ so we need not add the B_k, $k = 1, \ldots, j-1$), then we see by induction that B_j is a function of $X_0, T_1, T_2, \ldots$ Hence the underlying sample space can be taken to be the space of all sequences of successes and failures, with the usual product measure.

Suppose, more generally, that the player determined B_j by examining $X_0, \ldots, X_{j-1}$, and then "consulting" a chance device, e.g., a near-by roulette wheel. Then the sample space consisting of an infinite product of spaces, each of them a joint outcome of the roulette wheel and the latest trial, might be suitable. Such possibilities are included if we simply assume T_j, X_j and B_j are all random variables on *some* suitable sample space Ω.

When $B_j > X_{j-1}$, the player is betting more than he has. He is asking for credit. This is common in gambling casinos, in the stock market (buying on margin), in real estate (mortgages) and is not unrealistic.

When $B_j < 0$, the player is making a "negative" bet. To interpret this, we note that in our sequence of Bernoulli trials, or coin toss between *two* players, that what one wins, $X_j - X_0$, the other loses. To make a negative bet may be interpreted as "backing" the other side of the game, to taking the role of the "other" player.

In particular, the payoff B_jT_j from trial j may be written as $(-B_j)(-T_j)$. If $B_j \leqq 0$, then $-B_j \geqq 0$ and may be interpreted as a nonnegative bet by a player who succeeds when $-T_j = 1$, i.e., with probability q, and who fails with probability p. The $-T_j$ are independent so we have Bernoulli trials with success probabilities q, i.e., the other side of the game.

For simplicity we shall assume in what follows that $0 \leqq B_j \leqq X_{j-1}$, but we may wish at a future time to remove one or both of these limitations.

We also assume that B_j is independent of T_j, i.e., the amount bet on the jth outcome is independent of that outcome.

Definition: A betting strategy is a family $\{B_j\}$ such that $0 \leqq B_j \leqq X_{j-1}$, $j = 1, 2, \ldots$

Theorem 1: The betting strategies $B_j = X_{j-1}$ when $p > \frac{1}{2}$; $B_j = 0$, $p < \frac{1}{2}$; B_j arbitrary when $p = \frac{1}{2}$; are precisely the ones which maximize $E(X_j)$ for each j.

Proof: Since $X_n = X_o + \sum_{j=1}^{n} B_jT_j$, $E(X_n) = X_o + \sum_{j=1}^{n} E(B_jT_j) =$

$= X_0 + \sum_{j=1}^{n} (p-q) E(B_j)$. If $p - q = 0$, i.e., $p = \frac{1}{2}$, then $E(B_j)$ does not affect $E(X_n)$. If $p - q > 0$, i.e. $p > \frac{1}{2}$, then $E(B_j)$ should be maximized, i.e., $B_j = X_j$, to maximize $E(X_n)$. Similarly, if $p - q < 0$, i.e., $p < \frac{1}{2}$, then $E(B_j)$ should be minimized to maximize the jth term, i.e., $B_j = 0$. Clearly, these maxima are not attained with other choices for B_j. This establishes the theorem.

Remark. In the foregoing discussion, the Bernoulli trials and the T_j can be generalized, yielding a more general theorem. (The T_j become "payoff functions" that are not necessarily identically distributed; roulette is the classic example.) The particular case of blackjack is covered, for instance, by replacing p and q throughout by p_j and q_j for the respective probabilities that $T_j = 1$ or -1.

To maximize our expected gain we must bet our total resources at each trial. Thus if we lose once we are ruined, and the probability of this is $1 - p^n \to 1$ so maximizing expected gain is undesirable.

2. MINIMIZING THE PROBABILITY OF RUIN

Suppose instead that we play to minimize the probability of eventual ruin, where ruin occurs after the jth outcome if $X_j = 0$. If we impose no further restriction on B_j, then many strategies minimize the probability of ruin. For example, it suffices to choose $B_j < X_{j-1}/2$. The discreteness of money makes it realistic to assume $B_j \geq C > 0$, where C is a non-zero constant. We further restrict ourselves to the subclass of strategies where B_j equals C whenever $0 < X_{j-1} < a$, $B_j = 0$ if $X_{j-1} \leq 0$ or $X_{j-1} \geq a$, and C divides both $a - z$ and z, where we have set $z = X_0$. This lets us use the gambler's ruin formulae ([8], page 314).

Consider the gambler's ruin situation: $X_0 = z$, $B_j = 1$ if $0 < X_{j-1} < a$, $B_j = 0$ if $X_{j-1} = 0$ or $X_{j-1} = a$, a and z are integers. Let r be a positive number (necessarily rational) such that zr and ar are integers. Let $R(r)$ be the ruin probability when z and a are replaced by zr and ar, respectively. This is equivalent to betting r^{-1} units when $0 < X_{j-1} < a$, in the original problem.

We have $R(r) = (\theta^{ar} - \theta^{zr}) / (\theta^{ar} - 1)$, where $0 < p \neq \frac{1}{2}$ and $\theta = q/p$.

Theorem 2: (a) If $1 > p > \frac{1}{2}$, $R(r)$ is a strictly decreasing function of r. (b) If $0 < p < \frac{1}{2}$, $R(r)$ is a strictly increasing function of r.

Proof: Follows from Lemma 3 below.

Part (a) of the Theorem says that in a favorable game, the chance of ruin is decreased by decreasing stakes. Note that for $p > \frac{1}{2}$, i.e., $\theta < 1$, $\lim_{r\to\infty} R(r) = 0$, hence by making stakes sufficiently small, the chance of ruin can be made arbitrarily small.

Lemma 3. Let $a > z > 0$, $x > 0$. If $0 < \theta < 1$, then $f(x) = (\theta^{zx} - \theta^{ax}) / (1 - \theta^{ax})$ is strictly decreasing as x increases, $x > 0$. If $\theta > 1$, $f(x)$ is strictly increasing as x increases, $x > 0$.

Proof: Elementary calculations which we omit.

Theorem 2(a) shows that, at least in the limited subclass of strategies to which it applies, we minimize ruin by making a minimum bet on each trial.

In fact, this holds for a broader class of strategies:

Theorem 3′: If $1 > p > \frac{1}{2}$, the strategy $B_j = 1$ if $1 \leq z \leq a-1$, $B_j = 0$ otherwise (*timid play*), uniquely minimizes the probability of ruin among the strategies where B_j is an integer satisfying $1 \leq B_j \leq \min(z, a-z)$ if $1 \leq z \leq a-1$, $B_j = 0$ otherwise.

Proof: We first show that if timid play is optimal, then it is uniquely so. Let q_z be the probability of ruin, starting from z, under timid play. To establish uniqueness it suffices to show

$$q_z < pq_{z+k} + q\, q_{z-k},\ 2 \leqq k \leqq a\text{–}2,\ z\text{–}k \geqq 0,\ z + k \leqq a. \tag{1}$$

Using $q_z = (\theta^a\text{–}\theta^z) / (\theta^a\text{–}1)$ and simplifying (1), we find that it is equivalent to show

$$f(p) = p^{2k-1} + q^{2k-1} - p^{k-1} q^{k-1} > 0,\ 1/2 < p \leqq 1. \tag{2}$$

This follows at once from the observations $f(\frac{1}{2}) = 0$, $f(1) = 1$, and $f'(p) > 0$, $\frac{1}{2} < p \leqq 1$.

To show that timid play is optimal, let $Q(z) = 1\text{–}q_z$, and adopt the terminology of [6]. Then $Q(z)$ is the probability of success, both for z in our original game, and for z/n in a normalized game where the possible fortunes are $F = \{0, 1/a, 2/a, \ldots, z/a, \ldots, 1 = a/a\}$, and the betting units and limits are $1/a$ as large as before.

The establishment of (1) shows $Q(z)$ is excessive. But obviously $u(z) \leqq Q(z) \leqq U(z)$ so by [6, Theorem 2.12.3], $Q(z) = U(z)$.

Thus timid play is the one and only strategy in our class of strategies which minimizes the probability of ruin.

Remark: In [6] it is shown that bold play is optimal but not necessarily unique when $p < \frac{1}{2}$ (pages 2, 87ff, 101ff). If there is also a legal upper limit to bets, there may be more than one optimal strategy; whether bold play is one of them seems to be unknown (page 4). Betting systems which minimize the probability of ruin in certain favorable games are also discussed in [10].

The strategy which minimizes ruin has the unsatisfactory consequence that it also minimizes our expected gain. Some strategy is called for which is intermediate between minimizing ruin (and expectation) and maximizing expectation (assuring ruin). A remarkable solution, in a certain sense very close to best possible, was proposed in [13].

3. THE KELLY CRITERION

Consider Bernoulli trials with $1 > p > \frac{1}{2}$ and $B_j = f X_{j-1}$, where $0 \leqq f \leqq 1$ is a constant. (This is sometimes called "fixed fraction" or "proportional" betting.) Let S_n and F_n be the number of successes and failures, respectively, in n trials. Then

$$X_n = X_o (1 + f)^{S_n} (1 - f)^{F_n}.$$

Observe that $f = 0$ and $f = 1$ are uninteresting; we assume $0 < f < 1$. Note too that if $f < 1$, there is no chance that $X_n = 0$, ever. Hence ruin, in the sense of the gambler's ruin problem, cannot occur. We reinterpret "ruin" to mean that for each $\varepsilon > 0$, $\lim_n P[X_n \geqq \varepsilon] = 0$, and we shall see that this can occur. Note too that we are now assuming that capital is infinitely divisible. However, this assumption is not a serious problem in practical applications of the theory.

Remark: The min-max criterion of game theory is an inappropriate criterion in Bernoulli trials. If B_j is a positive integer for all j, the maximum loss, i.e. ruin, is always possible and all strategies have the same maximum possible loss, hence all are equivalent. If capital is infinitely divisible, ruin is as we redefined it, and we restrict ourselves to fixed fractions, then for an infinite series of trials the min-max criterion (suitably probabilitistically modified) considers all f with $1 \geqq f \geqq f_c$ equivalent and all f with $0 < f < f_c$ equivalent. It chooses the latter class. For a fixed number n of trials, smaller f are preferred over larger f. The criteria of minimizing ruin or of maximizing expectation likewise fail to make desirable distinctions between the fixed fraction strategies.

The quantity $\log [X_n / X_0]^{1/n} = (S_n / n) \log (1+f) + (F_n / n) \log (1-f)$ measures the rate of increase per trial. Since time is important it is plausible to in some sense maximize this. Kelly's choice [13] was to maximize $E \log [X_n / X_0]^{1/n} = p \log (1+f) + q \log (1-f) \equiv G(f)$, which we call the exponential rate of growth. The following theorems show the advantages of maximizing $G(f)$.

Theorem 4. If $1 > p > \frac{1}{2}$, $G(f)$ has a unique maximum at $f^* = p - q$, $0 < f^* < 1$, where $G(f^*) = p \log p + q \log q + \log 2 > 0$. There is a unique fraction $f_c > 0$ such that $G(f_c) = 0$, and f_c satisfies $f^* < f_c < 1$. Further, we have $G(f) > 0$, $0 < f < f_c$; $G(f) < 0$, $f > f_c$, with $G(f)$ strictly increasing, from 0 to $G(f^*)$, on $[0, f^*]$, and $G(f)$ strictly decreasing, from $G(f^*)$ to $-\infty$ on $[f^*, 1]$.

Theorem 5(*a*). If $G(f) > 0$, then $\lim_n X_n = \infty$ a.s., i.e., for each M, $P[\underline{\lim}_n X_n > M] = 1$.

(b) If $G(f) < 0$, then $\lim_n X_n = 0$ a.s., i.e., for each $\varepsilon > 0$, $P[\overline{\lim}_n X_n < \varepsilon] = 1$.

(c) If $G(f) = 0$, then $\overline{\lim}_n X_n = \infty$ a.s. and $\underline{\lim}_n X_n = 0$ a.s.

Thus for $0 < f < f_c$, the player's fortune will eventually permanently exceed any fixed bounds with probability one. For $f = f_c$ it will almost surely oscillate wildly between 0 and $+\infty$. If $f > f_c$, ruin is almost sure.

Proof: (a) By the Borel strong law ([17], page 19), $\lim_n \log [X_n / X_0]^{1/n} = G(f) > 0$ with probability 1. Hence, a.s., for $\omega \in \Omega$, where Ω is the space of all sequences of Bernoulli trials, there exists $N(\omega)$ such that for $n \geq N(\omega)$,
$\log [X_n / X_0]^{1/n} \geq G(f)/2 > 0$.
But then $X_n \geq X_0 e^{n G(f)/2}$ for $n \geq N(\omega)$ so $X_n \nearrow \infty$.

(b) The proof is similar to part (a).

(c) We use the fact that, given any M, $\overline{\lim}_n S_n \geq np + M + 1$ and $\underline{\lim}_n S_n \leq np - M - 1$. Then if $S_n \geq np + M$, $\log [X_n / X_0]^{1/n} \geq \frac{np + M}{n} \log (1+f) + \frac{n - (np + M)}{n} \log (1-f) = G(f) + \frac{M}{n} \log \frac{1+f}{1-f} = \frac{M}{n} \log \frac{1+f}{1-f}$, whence $X_n \geq X_0 \left(\frac{1+f}{1-f}\right)^M$. Since $S_n \geq np + M$ infinitely often, a.s., then $\overline{\lim}_n X_n \geq X_0 \left(\frac{1+f}{1-f}\right)^M$ a.s. Since the right side may be chosen arbitrarily large, $\overline{\lim}_n X_n = \infty$ a.s.

The proof that $\underline{\lim}_n X_n = 0$ a.s. is similar.

Theorem 6: If $G(f_1) > G(f_2)$, then $\lim_n X_n(f_1) / X_n(f_2) = \infty$ a.s.

Proof: $\log [X_n(f_1) / X_0]^{1/n} - \log [X_n(f_2) / X_0]^{1/n}$
$= \log [X_n(f_1) / X_n(f_2)]^{1/n} = \frac{S_n}{n} \log \left(\frac{1+f_1}{1+f_2}\right) + \frac{F_n}{n} \log \left(\frac{1-f_1}{1-f_2}\right)$. Therefore, by the Borel strong law of large numbers, $\lim_n \log [X_n(f_1) / X_n(f_2)] \to G(f_1) - G(f_2) > 0$ with probability 1. Now proceed as in the proof of Theorem 5(a).

In particular, we see that if one player uses f^* and another, *betting on the same favor-*

able situation, uses any other fixed fraction strategy f, then $\lim_n X_n(f^*)/X_n(f) = \infty$ with probability 1. This is one of the important justifications of the criterion "bet to maximize $E \log X_n$."

Bellman and Kalaba ([2], pages 200–201) show that f^* not only maximizes $E \log X_n$ within the class of all fixed fraction betting strategies but in the class of "all" betting strategies.

This also is a consequence of the following theorem, part of which was suggested by a conversation with J. Holladay. Consider a series of independent trials in which the return on one unit bet on the ith outcome is the random variable Q_i. Then $X_n = \prod_{i=1}^{n} (X_i/X_{i-1})$ and $E \log X_n = \sum_{i=1}^{n} E \log (X_i/X_{i-1})$. We have $X_i = X_{i-1} + B_i Q_i$ and $X_i/X_{i-1} = 1 + (B_i/X_{i-1}) Q_i$. Thus each term is of the form $E \log (1 + F_i Q_i)$ where the random variable F_i depends only on the first i–1 trials, Q_i depends only on the ith trial, and hence F_i and Q_i are independent. We are free to choose the F_i to maximize $E \log (1 + F_i Q_i)$, subject to the constraint $0 \leq F_i \leq 1$.

Theorem 7: If for each i there is an f_i, $0 < f_i < 1$, such that $E \log (1 + f_i Q_i)$ is defined and positive, then for each i there is a number f_i^* such that $E \log (1 + F_i Q_i)$ attains its unique maximum for $F_i = f_i^*$ a.s. To avoid trivialities we assume $Q_i \neq 0$ a.s., each i.

Proof: It follows that the domain of definition of $E \log (1 + f_i Q_i)$ is an interval $[0, a_i)$ or $[0, a_i]$, where $a_i = \min (1, b_i)$ and $b_i = \sup \{f_i : f_i Q_i > 0 \text{ a.s.}\} > 0$. Since the second derivative with respect to f_i of $E \log (1 + f_i Q_i)$ is $-E(Q_i^2/(1 + f_i Q_i)^2)$, which is defined and negative, any maximum of $E \log (1 + f_i Q_i)$ is unique. The function is continuous on its domain so if it is defined at a_i, there is a maximum. If it is not defined at a_i, then $\lim_{f_i \uparrow a_i} E \log (1 + f_i Q_i) = -\infty$ so again there is a maximum.

By the independence of F_i and Q_i, we can consider $F_i(s_1)$ and $Q_i(s_2)$ as functions on a product measure space $S_1 \times S_2$. Then

$E \log (1 + F_i Q_i) = \int_{S_1} \int_{S_2} \log (1 + F_i(s_1) Q_i(s_2)) = E\, E \log (1 + F_i(s_1) Q_i)$

$\leq E \log (1 + f_i^* Q_i)$ with equality if and only if $E \log (1 + F_i(s_1) Q_i) = E \log (1 + f_i^* Q_i)$ a.s., which is equivalent to $f_i^* Q_i = F_i(s_1) Q_i$ a.s., and by the independence this means either $f_i^* = F_i$ a.s. or $Q_i = 0$ a.s. hence $f_i^* = F_i$ a.s., and the theorem is established.

We see in particular from the preceding theorem that for Bernoulli trials with success probability p_i on the ith trial and $1 > p_i > \frac{1}{2}$, $E \log X_n$ is maximized by simply choosing on each trial the fraction $f_i^* = p_i - q_i$ which maximizes $E \log (1 + f_i Q_i)$.

4. THE ADVANTAGES OF MAXIMIZING E LOG X_n

The desirability of maximizing $E \log X_n$ was established in a fairly general setting by Breiman [3]. Consider repeated independent trials with finitely many outcomes $I = \{1, \ldots, s\}$ for each trial. Let $P(i) = p_i$, $i = 1, \ldots, s$, and suppose that $\{A_1, \ldots, A_r\}$ is a collection of (betting) subsets of I, that each i is in some A_k, and that payoff odds o_k correspond to the A_k. We bet amounts $B_1, \ldots, B_r$ on the respective A_k and if outcome i occurs, we receive $\sum B_j o_j$ where the sum is over $\{j : i \in A_j\}$. We make the convention that $A_1 = I$ and $o_1 = 1$, which allows us to hold part of our fortune in reserve by simply betting it on A_1. We have, in effect, a generalized roulette game.

Roulette and the wheel of fortune, as described in Part I, are covered directly by Breiman's theory.

The theory easily extends to independent trials with finitely many outcomes which are not identically distributed but which are a mix of finitely many distinct distributions, each occurring on a given trial with specified probabilities. The theory so extended applies to Blackjack and Nevada Baccarat, as described in Part I.

Breiman calls a game (i.e., a series of such trials) favorable if there is a gambling strategy such that $X_n \to \infty$ a.s. Thus the infinite divisibility of capital is tacitly assumed. However, this is not a serious limitation of the theory. If the probability is "negligible" that the player's capital will at some time be "small," then the theory based on the assumption that capital is infinitely divisible applies to good approximation when the player's capital is discrete. This problem is considered for Nevada Baccarat in [24], pages 319 and 321.

Breiman establishes the following about strategies which maximize $E \log X_n$.

1. Allowing arbitrary strategies, there is a fixed fraction strategy $B_i = (f_1, \ldots, f_r)$ which maximizes $E \log X_n$.

2. If two players bet on the same game, one using a strategy Λ^* which maximizes $E \log X_n$ and the other using an "essentially different" strategy Λ, then $\lim_n X_n(\Lambda^*) / X_n(\Lambda) \to \infty$ a.s.

3. The expected time to reach a fixed preassigned goal x is, asymptotically as x increases, least with a strategy which maximizes $E \log X_n$.

Thus strategies which maximize $E \log X_n$ are (asymptotically) best by two reasonable criteria.

5. A STOCK MARKET EXAMPLE

Though in practice there are only finitely many outcomes of a bet in the stock market, it is technically convenient to approximate the finite distributions by discrete countably infinite distributions or by continuous distributions. In fact it is generally difficult not to do this. The additional hypotheses and difficulties which occur are, from the practical point of view, artificial consequences of the technique. Hence the new theory must preserve the conclusions of the finite theory so again we apportion our resources to maximize $E \log X_n$.

As a first example, consider the following stock market investment. It was the first to catch our interest, and was based on a tip from a company insider.

Suppose a certain stock now sells at 20 and that the anticipated price of the stock in one year is uniformly distributed on the interval [15, 35]. We first compute f^* and $G(f^*)$, assuming the stock is purchased and fully paid for now, and sold in one year. The purchase and selling fees have been included in the price. Thus, the outcome of this gamble, per unit bet, is described by $dF(s) = C_{(-\frac{1}{4}, \frac{3}{4})}(s)\, ds$, where F is the associated probability distribution and $C_A(s)$ is 1 for s in A and 0 for s not in A.

The mean m of F is $\frac{1}{4} > 0$. Also

$$G(f) = \int_{-\frac{1}{4}}^{\frac{3}{4}} \log_2(1+fs)\, ds,\quad G'(f) = \int_{-\frac{1}{4}}^{\frac{3}{4}} \frac{s \log_2 e}{1+fs}\, ds, \text{ and } \lim_{f \uparrow 4} G'(f) = -\infty.$$

Therefore Theorem 8 below applies and there is a unique f^* such that $0 < f^* < 4$ and $G'(f^*) = 0$. To obtain f^*, it suffices to solve

$$h(f) = 0 \text{ where } h(f) = fG'(f)/\log_2 e = \int_{-\frac{1}{4}}^{\frac{3}{4}} \frac{fs\, ds}{1+fs} = \int_{-\frac{1}{4}}^{\frac{3}{4}} ds - \int_{-\frac{1}{4}}^{\frac{3}{4}} \frac{ds}{1+fs} =$$

$$= 1 - (1/f)\log_e(1+fs)\Big|_{-\frac{1}{4}}^{\frac{3}{4}} \text{ which reduces to } 1 - (1/f)\log_e \frac{1+\frac{3}{4}f}{1-\frac{1}{4}f} = h(f).$$

Now $h(f)$ has the same sign and root as $G'(f)$ on (0,4). Since $h(3) = 1 - \frac{1}{3} \log_e 13 > 0$, $G'(f) > 0$ for $0 \leqq f \leqq 3$. Therefore $3 < f^* < 4$; calculation yields $f^* = 3.60^-$.

Thus if the maximum fraction of current capital which can be bet is 1, we should bet all our capital. However, if margin buying is allowed, we should (consistent with our ability to cover later) be willing to bet as much as possible, up to a fraction f^* which is 3.6 times our current capital.

The mathematical expectation for buying outright is 0.25 V_0 if buying on margin is excluded and 0.90 V_0 if unlimited buying on margin is permitted, and additional coverings can be made later as required, and we bet $f^* = 3.60$ of our current capital.

Integration yields

$G(f) = [(\log_2 e)/f]\{(1+3f/4)[\ln(1+3f/4)-1]-(1-f/4)[\ln(1-f/4)-1]\}$ from which we find $G(1) = 0.28$ and $G(3.60) = .59$.

Next, we compute f^* and $G(f^*)$, assuming that calls are purchased for 2 points per share. Thus the outcome of this gamble per unit bet is described by the probability distribution F with mass 5/20 at -1, and $dF(s) = 2/20$ if $-1 < s < 6.5$.

The mean $m = 1.8125 > 0$. Also $G(f) = (5/20)\log_2(1-f) + (2/20)\int_{-1}^{6.5} \log_2 (1+fs)\,ds$ and $G'(f) = \dfrac{-(5/20)\log_2 e}{1-f} + (2/20)\int_{-1}^{6.5} \dfrac{s \log_2 e}{1+fs}\,ds$, from which it is clear that $\lim_{f\uparrow 1} G'(f) = -\infty$. Therefore, again by Theorem 8 below, there is a unique f^* such that $G'(f^*) = 0$ and $0 < f^* < 1$.

It suffices to solve $h(f) = 0$ where $h(f) = 20G'(f)/\log_2 e$ $= \dfrac{-5}{1-f} + \dfrac{2}{f}\left\{7.5 - \dfrac{1}{f}\log_e\left(\dfrac{1+6.5f}{1-f}\right)\right\}$. We find $f^* = 0.57$. The mathematical expectation of the call purchase process is $1.8125 f^* V_0$ or about $1.03\ V_0$.

Integration yields

$G(f) = (\frac{1}{4})\log_2(1-f) + (\ln_2 e/10f)(1+6.5f)[\ln(1+6.5f)-1]-(1-f)[\ln(1-f)-1]$. We find $G(0.57) = 0.55$.

Thus we have the interesting result that the expectation from buying calls is higher than from buying on unlimited margin but that the growth coefficient is higher from buying on unlimited margin. Our criterion selects the latter investment.

For buying on margin $G(3^-) = .55$ so our criterion selects buying on margin if the margin requirement is less than $\frac{1}{3}^-$ and buying calls, if possible, if the margin requirement exceeds $\frac{1}{3}^-$.

In the preceding example we needed the following theorem to establish the uniqueness of f^*. We define $a = \sup\{t : F(-\infty, t) = 0\}$ and note that if $1 + fa > 0$ and the integral $G(f) = \int_a^\infty \log_2(1+fs)\,dF(s)$ is defined, then $G'(f) = \int_a^\infty \dfrac{s \log_2 e}{1+fs}\,dF(s)$. (See, e.g. [17], page 126).

Theorem 8: The function $G'(f) = \int_a^\infty \dfrac{s\log_2 e}{1+fs}\,dF(s)$ is monotone strictly decreasing on $[0, -1/a)$. If the mean $m = \int_a^\infty s\,dF(s) > 0$, then the equation $G'(f) = \int_a^\infty \dfrac{s\log_2 e}{1+fs}\,dF(s) = 0$ has exactly one solution f^* in the interval $(0, -1/a)$ iff $\lim_{f\nearrow -1/a} G'(f) < 0$. In this event, $G(f)$ is monotonely strictly increasing for f in $[f^*, -1/a)$.

Proof: If $0 < f_1 < f_2 < -1/a$, $\dfrac{s}{1+f_1 s} > \dfrac{s}{1+f_2 s}$ $(0 \neq s \geqq a)$ so $G'(f_1) > G'(f_2)$.

From this and the right-hand continuity of $G'(f)$ at 0, G' is monotone strictly decreasing on $[0, -1/a)$. By hypothesis $G'(0) = m \log_2 e > 0$. Therefore, from the continuity of $G'(f)$ on $[0, -1/a)$, $G'(f)$ attains all values t on the interval $\lim_{f \nearrow -1/a} G'(f) < t \leqq G'(0)$ exactly once. Thus there is exactly one solution f^* in $(0, -1/a)$ iff $\lim_{f \nearrow -1/a} G'(f) < 0$.

The description of $G(f)$ is now evident.

6. WARRANT HEDGING

We next apply the criterion of maximizing $E \log X_n$ to the warrant hedge described in Part I. With the notation of Part I, and an assumed mix of 1, the gain X from a one unit bet is

$$X = (s_f - s_0 + w_0) / (\alpha s_0 + \beta w_0),\ s_f \leqq 1, \text{ and}$$
$$X = (w_0 + 1 - s_0) / (\alpha s_0 + \beta w_0),\ s_f > 1.$$

We wish to maximize the exponential rate of growth $G(f)$, given by $G(f) = E \log (1 + fX)$.

It can be shown that the situation is essentially the same as in Theorem 8 and that this depends on the a.s. boundedness of X; we have in fact a.s. $\sup X = (w_0 + 1 - s_0) / (\alpha s_0 + \beta w_0)$ and a.s. $\inf X = -(s_0 - w_0) / (\alpha s_0 + \beta w_0)$. Thus f^* can be computed when the mix is 1, though the details are tedious.

When the mix is greater than 1, more serious difficulties appear. The payoff function X has a.s. $\inf X = -\infty$ and a.s. $\sup X < \infty$. This means that, no matter what fraction $f > 0$ of our unit capital is bet, there is positive probability of losing at least the entire unit. Thus any bet is rejected! Yet this is unrealistic. We now find out what is wrong.

First, the assumption that arbitrarily large losses have positive probability of occurrence is not realistic. (a) The broker will automatically act to liquidate the position before the equity is lost. (b) The strategies for investing in hedges automatically lead to liquidating the position after the common is substantially above exercise price.

There is, then, a maximum imposed on X by practice but it is not easy in practice to specify this maximum. Further, this maximum will, in general, be a random variable (a.s. bounded, however) which is a function of the individual's investment strategy. It is not easy to determine the consequent probability distribution of s_f, yet this is required to calculate $E \log (1 + fX)$.

More generally, we might consider an individual's lifetime sequence of bets of various kinds. It is plausible to assume that $X_n = 0$ only upon the death of the individual, for although the individual may have no cash equity at a given instant, he does have a cash "worth", based on his future income, serendipity, etc., and this should be included in X_n. This is true even of a (Billie Sol Estes) bettor who loses more than he owns. The subtlety here, then, is that the accountant's figure for net assets (plus or minus) is not an accurate figure for X_n as X_n decreases below small positive amounts.

One can also object to X_n at death being assigned the value 0, by arguing that the chance of death in a time interval always has a small positive probability, thus making $E \log X_n = -\infty$ always. Also, individuals when choosing between two alternatives each involving a low probability of death generally do not meticulously select the safer alternative (e.g., air travel versus train travel). Thus death should really be treated as an event with a large but finite negative value.

Another common objection to $E \log X_n$ as a measure of "utility" is that, like all such measures which are not a.s. bounded, it allows the St. Petersburg paradox.

The foregoing objections to $E \log X_n$ only arise when we leave the case of finitely many outcomes. We say that these are artificial technical difficulties which can all be removed in the cases of practical importance. This may be tedious, as it is for the warrant hedge, so we defer such matters for a subsequent paper.

7. PORTFOLIO SELECTION USING $E \log X$

The Breiman results were obtained for repeated independent trials with finitely many outcomes and finitely many ways to apportion our capital (amongst finitely many betting sets). The results extend, as we have remarked, to independent trials which are a mix of finitely many differently distributed trials (i.e., finitely many outcomes and betting sets) provided that as n tends to infinity, the number of trials with each distribution also tends to infinity.

There are significant real world situations, such as the selection and continuous revision of a portfolio of securities, to which this extended theory does not generally apply. A difficulty which we have already discussed is that it may be technically convenient to introduce continuously distributed and possibly unbounded payoffs, but now generalized to the apportionment of capital among a finite number of alternatives, rather than just betting a fraction on one alternative. Another problem is that the sequence of betting situations may change so that no two are ever the same. Further difficulties arise when we consider the possible dependence of trials. Still other problems appear when we consider that in the real world the spectrum of situations is changing continuously and that a potentially continuous portfolio revision is part of an optimal approach. (Actually, because of the transactions costs which occur in practice, portfolio revision is likely to occur in discrete steps.)

The extent to which Breiman's conclusions for the finite case can be generalized in these directions will be considered subsequently. For now we simply remark that the possible generalizations promise to be adequate for the real world problems of portfolio selection.

Assuming this to be the case, we shall see in the next section that economists and others now have for the first time an accurate guide for portfolio selection and revision.

8. THE KELLY CRITERION AND DEFICIENCIES IN THE MARKOWITZ THEORY OF PORTFOLIO SELECTION

How to apportion funds among investments has endlessly puzzled economists and decision-makers. The literature was noted for its lack of instruction in such matters. When Markowitz' work on portfolio selection appeared, first in articles and later in the monograph [18], it became the standard reference.

Markowitz considers situations in which there are r alternative and, in general correlated, investments, with the gain per unit invested of $X_1, \ldots, X_r$, respectively. (It is so much more dignified to call bets investments; we shall try to remember to do this in this section.) One of the investments is, of course, cash. The gain is given by $X_k = 0$ a.s.

To select a portfolio is to apportion our resources so that f_i is placed in the ith investment. Markowitz' basic idea is that a portfolio is better if it has higher expectation and at least as small a variance or if it has at least as great an expectation and has a lower variance. If two portfolios have the same expectation and variance, neither is preferable. As the f_i range over all possible admissible values, the set of portfolios is generated. Typically the assumptions on the f_i are $\Sigma f_i = 1$, and $f_i \geq 0$ for $i = 1, \ldots, r$.

If a portfolio has the property that no other portfolio in the set is preferable, then it is called efficient. Markowitz says that the investor should always choose an efficient portfolio. Which efficient portfolio to choose depends on factors outside the theory, such as the investor's "needs".

The Markowitz theory has the obvious deficiency that if E_i and σ_i^2, $i = 1, 2$, are the expectation and variance of portfolios 1 and 2, then if $E_1 < E_2$ and $\sigma_1^2 < \sigma_2^2$, the theory cannot choose between the portfolios. Yet there are obvious instances where "everyone" will choose the second portfolio over the first, such as when $F_1(x) < F_2(x)$ for all x. Specifically, let X_1 be distributed uniformly on $[1, 3]$, let X_2 be uniformly distributed on $[10, 100]$ and let $X_3 = 0$ a.s. represent the possibility of holding some of our resources in cash. Suppose X_1, X_2, and X_3 are independent. Then $E\Sigma f_i X_i = 2f_1 + 55f_2$ and $\sigma^2 \Sigma f_i X_i = \Sigma f_i^2 \sigma_i^2 = f_1^2/3 + 675 f_2^2$. All cash, or $f_3 = 1$, is an efficient portfolio since this is the unique portfolio with zero expectation. The portfolio $f_2 = 1$ also is efficient since this is the unique portfolio with greatest expectation. There are, in fact, infinitely many efficient portfolios. (They lie on a curve in the f_1, f_2 plane connecting $(0, 0)$ and $(0, 1)$.) The theory doesn't tell us which is best, yet $f_2 = 1$ is clearly preferable to any alternative.

In the case where there are the two alternatives $X_1 = 0$ a.s. (cash) and X_2 with $E_2 > 0$ and $\sigma_2 > 0$, all portfolios are efficient and Markowitz' theory gives no information on which to choose. The Kelly criterion tells us to choose f_2 to maximize $E \log(1 + f_2 X_2)$ and we know further from the theory of the Kelly criterion why this choice is good. As we have seen, repeated trials of such an investment with f_2 greater than the fraction f_c will lead to ruin a.s.

Remark: This incompleteness of Markowitz' theory is understandable since he only uses probability information about first and second moments. We note though that the examples he gives, and the real world applications, generally assume that more detailed structure is known. Hence, it is reasonable that the criterion $E \log X_n$, which does use higher moment information, can provide a sharper theory.

Next consider those two-point probability distributions with masses m_i located at x_i, $i = 1, 2$, and with mean and variance 1. These are indistinguishable by Markowitz' criterion. A calculation shows, however, that for X_1 defined by $x_1 = -1$, $x_2 = 3/2$, $m_1 = 1/5$, $m_2 = 4/5$, the optimal fraction f_1^* is $\frac{2}{3}$ and $G(f_1^*)$ is $-(1/5)\log 3 + (4/5)\log 2$. For X_2 defined by $x_1 = -2$, $x_2 = 4/3$, $m_1 = 1/10$, $m_2 = 9/10$, we have $f_2^* = \frac{3}{8}$ and $G(f_2^*) = -(1/10)\log 4 + (9/10)\log(3/2)$, which is smaller than $G(f_1^*)$. Hence if $X_{n,1}^*$ is the fortune after n repeated independent trials of an investor who invests f_1^* in X_1 at each trial and $X_{n,2}$ is the fortune after n trials of an investor who invests in any manner whatsoever in X_2 at each trial, we have $\lim X_{n,1}^* / X_{n,2} = \infty$ a.s.

As a final example, suppose we are to apportion our resources between the foregoing X_1 and X_2, which we now suppose to be independent, and cash, represented by X_3. We impose the constraints $f_i \geqq 0$, $i = 1, 2, 3$; $f_1 + f_2 + f_3 = 1$, and $f_1 + 2f_2 \leqq 1$. The latter constraint prevents investments where our losses exceed our total resources. (The analysis and conclusion are essentially the same without this constraint.) The admissible portfolios are represented by the closed triangular region of the positive quadrant bounded by the axes and the line $f_1 + 2f_2 = 1$.

We have $E\Sigma f_i X_i = f_1 + f_2$ and, because of the independence of X_1 and X_2, $\sigma^2 \Sigma f_i X_i = f_1^2 + f_2^2$. The efficient portfolios are the points of the f_1, f_2 plane on the two closed line segments joining $(\frac{1}{3}, \frac{1}{3})$ to $(0, 0)$ and to $(1, 0)$.

The function $E \log(1 + f_1 X_1 + f_2 X_2) \equiv G(f_1, f_2)$ is given by $50\, G(f_1, f_2) =$

$36 \log(1 + 3f_1/2 + 4f_2/3) + 4 \log(1 + 3f_1/2 - 2f_2) + 9 \log(1 - f_1 + + 4f_2/3) + \log(1 - f_1 - 2f_2)$. This function is undefined on the line joining $(0, 1/3)$ and $(1, 0)$. It is defined and continuous elsewhere on the triangle of portfolios and as (f_1, f_2) tends to the segment from this triangle, $G(f_1, f_2) \to -\infty$. It follows (by the continuity) that $G(f_1, f_2)$ attains an absolute maximum in the region of the triangle where it is defined. We also know that any such maximum is positive. It follows that, if an efficient portfolio maximizes $G(f_1, f_2)$, then it must be a portfolio from the interior of the segment joining $(0, 0)$ and $(1/3, 1/3)$. Hence the coordinates must simultaneously satisfy the equations $\partial G(f_1, f_2) / \partial f_1 = 0$ and $\partial G(f_1, f_2) / \partial f_2 = 0$. (We note that in repeated independent trials where the investor selects an efficient portfolio from the segment joining $(1/3, 1/3)$ to $(1, 0)$, he will be ruined with probability one.)

Setting $f_1 = f_2 = t$ in the equations $\partial G / \partial f_1 = 0$ and $\partial G / \partial f_2 = 0$ and attempting to solve simultaneously yields, upon elimination between the two equations of the last of the four fractions, the necessary condition $-2796 + 376t + 111t^2 = 0$. Since this is negative at $t = 0$ and $t = 1$, there are no roots in the interval $0 < t < 1/3$. Hence no efficient portfolio maximizes $G(f_1, f_2)$.

We conclude that if $X^*_{n,1}$ is the fortune after n trials of a player who bets to maximize $G(f_1, f_2)$ on each trial, and $X_{n,2}$ is the fortune of a player who chooses any efficient portfolio on each trial, then $\lim X^*_{n,1} / X_{n,2} = \infty$ a.s. Furthermore, the Kelly investor will reach a fixed goal x in less time, asymptotically as $x \to \infty$, than a Markowitz investor.

The Kelly criterion should replace the Markowitz criterion as the guide to portfolio selection.

REFERENCES

[1] Baldwin, Cantey, Maisel, and McDermott (1956). The optimum strategy in blackjack, *J. Amer. Statist. Assoc.*, 51, 429–439.

[2] Bellman, R., Kalaba, R. (Sept. 1957). On the role of dynamic programming in statistical communication theory. *IRE Trans. of the professional group on information theory*, IT-3 no. 3, 197–203.

[3] Breiman, L. (1961). Optimal gambling systems for favorable games. *Fourth Berkeley Symposium on probability and statistics*, 1, 65–78.

[4] Coddington, E. A., Levinson, N. (1955). *Theory of Ordinary Differential Equations*. New York, McGraw-Hill.

[5] Cootner, P. H., editor (1964). *The Random Character of Stock Market Prices*. The M.I.T. press.

[6] Dubins, L., Savage, L. (1965). *How to Gamble if You Must*. New York, McGraw-Hill.

[7] Dunnington, G. Waldo (1955). *Carl Friedrich Gauss, Titan of Science*. New York, Hafner.

[8] Feller, W. (1957). *An Introduction to Probability Theory and Its Applications, Vol. I, Revised*. New York, Wiley.

[9] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications, Vol. II*. New York, Wiley.

[10] Ferguson, T. S. (1965). Betting systems which minimize the probability of ruin. *J. Soc. Indust. Appl. Math.*, 13, no. 3.

[11] Foster, F. G. (1964). A computer technique for game-theoretic problems. I. Chemin-de-fer analyzed. *Comput. J.*, 1, 124–130.

[12] Kassouf, Sheen T. (1965). *A Theory and an Econometric Model for Common Stock Purchase Warrants*. Ph. D. Thesis, Columbia University, New York.

[13] Kelly, J. L. (1956). A new interpretation of information rate. *Bell System Technical Journal*, 35, 917–926.

[14] Kemeny, J. G., Snell, J. L. (1957). Game theoretic solution of baccarat, *Amer. Math. Monthly*, 114, no. 7, 465–9.

[15] Kendall, M. G., Murchland, J. D. Statistical aspects of the legality of gambling, *J. Roy. Statist. Soc. Ser. A*. To be published.

[16] Levy, R. A. (April, 1966). *An Evaluation of Selected Applications of Stock Market Timing Techniques, Trading Tactics and Trend Analysis.* Ph. D. Thesis, The American University, Washington, D.C.
[17] Loève, M. (1963). *Probability Theory, Third Edition.* Princeton, Van Nostrand.
[18] Markowitz, H. (1959). *Portfolio Selection.* New York, Wiley.
[19] Pearson, K. (Jan.-June 1894). Science and Monte Carlo. *Fortnightly Review*, 61.
[20] Poincaré, H. *Science and Method*, an unabridged edition of the first English translation, by Francis Maitland. Dover.
[21] Poincaré, H. (1905). *Science and Hypothesis*, an unabridged republication of the first English translation. Dover, 1952.
[22] Thorp, E. (1966). *Beat the Dealer*, revised. New York, Random House.
[23] Thorp, E., Kassouf, S. (1967). *Beat the Market.* New York, Random House.
[24] Thorp, E., Walden, W. (1966). A winning bet in Nevada Baccarat. *J. Amer. Statist. Assoc.*, 61 Part I, 313–328.
[25] Wilson, A. (1965). *The Casino Gambler's Guide.* New York, Harper and Row.

RESUME

Mises optimales dans le cas de "jeux favorables"

Au cours de la dernière décade on a constaté que le joueur pouvait avoir l'avantage dans certains jeux de hasard. On verra que le "blackjack", la mise latérale au Baccara – tel qu'il est joué dans le Névada – la roulette et la "roue de la fortune", peuvent tous offrir au joueur une espérance de gain positive. La Bourse a beaucoup de traits communs avec ces jeux de hasard [5]. Elle offre des situations particulières avec des gains attendus allant au-delà d'un taux annuel de 25% [23].

Dès que la théorie particulière d'un jeu a été utilisée pour identifier des situations favorables, se pose le problème de savoir comment répartir au mieux nos ressources. Parallèlement à la découverte de situations favorables dans certains jeux, les grandes lignes d'une théorie mathématique générale pour exploiter ces opportunités s'est développée [2. 3. 10. 13.].

On décrira d'abord les jeux favorables mentionnés ci-dessus: ce sont ceux que l'auteur connait le mieux. On discutera ensuite la théorie mathématique générale, telle qu'elle s'est développée jusqu'à maintenant, et son application à ces jeux. Une connaissance détaillée d'un jeu particulier n'est pas nécessaire pour suivre l'explication. Chaque discussion portant un jeu favorable dans la partie I est suivie d'un résumé donnant les probabilités correspondantes. Ces résumés sont suffisants pour la discussion de la partie II de sorte qu'un lecteur qui n'a aucun intérêt dans un jeu particulier peut passer directement au résumé.

Des références sont données pour ceux qui désirent étudier certains jeux en détail. Pour l'instant, "jeu favorable" veut dire, jeu dans lequel la stratégie est telle que $P(\lim S_n = \infty) > 0$ où S_n est le capital du joueur après n essais.