Nontransitive Dice with Equal Means*

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Abstract. We analyze a game in which two players choose in turn and roll an n-sided die, each player having his choice of numbering of the faces, subject to certain constraints. The player who rolls the larger number wins. We say die B "dominates" die A if $P[B > A] > P[A > B]$. The nontransitivity of this dominance relation for dice has been studied by several authors. We analyze the case of n-sided dice for which the faces are equally likely to be uppermost when rolled, the values of the numbers on the faces sum to $n(n+1)/2$, and these values are chosen with replacement from the integers 1 through n. Our analysis shows that in contests that are scored numerically, the relation "A dominates B" is nontransitive even though all participants have the same expected score. We illustrate this with an example of m money managers ($m \geq 3$) who all have the same mean performance but for each $i = 0, 1, \ldots, m-1$, manager i is expected to outperform manager $i+1$ (mod m).

1 Introduction

Suppose you and an opponent each have a six-sided die, which is labeled using positive integers between 1 and 6 (inclusive), subject to the constraint that the sum is 21. Call such a die *admissible*. You each roll and, if the uppermost faces are unequal, the player with the larger integer wins. Assume each face is equally likely to be uppermost.

How shall you choose your die? Is there an "optimal strategy"? Do you want to choose your die first or second (assuming choosing the same die as your opponent is not an option)? What is the value of the game? We show, below, that the "standard" die guarantees a win on average 1/2 the time for any player who chooses it, against all choices of a die by the other player. For any die not the standard die, there is another die that is "better," in the sense that choosing it gives a probability of winning more than half the time (after excluding ties.) This is expressed by Theorem 1, which is a corollary of Theorem 2, proved below. Dice with n sides are also considered. Further elaboration of these ideas follows after the theorems, presented below.

* 2000 MSC: primary 91A60; secondary 62C20, 91A05.

Theorem 1. *For any die A not the "standard" die, there is another die B that "dominates" it, in the sense that $P[B > A] > P[A > B]$.*

Note that from this result we can construct sets of so-called nontransitive dice, as follows: Write $B \succ A$ if B dominates A. There are a finite number m ($m = 31$, actually) of nonstandard dice. Starting with any nonstandard die A_1, repeated application of the theorem yields $A_{m+1} \succ A_m \succ \cdots \succ A_2 \succ A_1$. By the pigeonhole principle $A_{m+1} = A_k$ for some k, $1 \le k \le m$. For no die A is it the case that $A \succ A$; and for no dice A and B is it the case that $A \succ B$ and $B \succ A$. It follows then that the cycle $A_{m+1} \succ A_m \succ \cdots \succ A_k = A_{m+1}$ contains at least three distinct dice, so the relation "$\succ$" violates transitivity.

2 Dice with n sides

We shall investigate the mathematical structure of the relationship "dominates" more generally on the set of n-sided dice with faces chosen (with repetition) from the first n positive integers, constrained to sum, like the "standard" die, to $n(n+1)/2$. For this general case we again call such dice *admissible*. The "standard" n-sided die, Z_n, has faces $(1, 2, \ldots, n)$. We shall drop the subscript n when its value is clear. Among other results, we shall show that Theorem 1 holds for $n \ge 4$.

Nontransitive dice have been discussed previously (e.g., Blyth 1972, Gardner 1983, Tenney and Foster 1976) where values of the faces could range freely over the positive integers and there was no limit on their sum. They arise naturally as an example of nontransitive voting preference leading to voting "paradoxes" (e.g., Saari 1995, 2001, Savage 1994, Smith 2001). Our approach is to explore nontransitivity when we constrain the dice to be "as much like the standard die as possible," namely, that their means equal the mean of the standard die, and the faces are chosen from the same set of numbers as the standard die. The increased constraints create more structure and more specific results.

We start by looking at the trivial cases: $n = 1, 2, 3$. There is only one 1-sided die, which can be modeled as a sphere with the label "1." There is only one 2-sided die, which can be modeled as a coin (with the rim rounded so it cannot end up "on edge"). For $n \ge 3$ we can use a prism with cross section a regular n-sided polygon and ends trimmed to convexly fit a hemisphere whose equator circumscribes the prism and is perpendicular to its axis. This prevents the prism from standing on end. The prism is tossed in a "random" way and lands on a horizontal plane surface. Poundstone (1992) tells the story of how RAND manufactured a stubby cylinder designed to fall with probability 1/3 on each face and on the edge. When von Neumann was confronted with the "coin," he thought briefly and correctly announced its proportions. For von Neumann's solution see Mosteller (1965).

For $n = 3$ we have two choices, the standard die Z_3, with faces labeled $(1, 2, 3)$, and the "symmetric" die S with faces $(2, 2, 2)$. It is easily seen in this case that $P[Z > S] = P[S > Z]$, which leads us to a useful general fact.

Define $Q(A, B) = n^2\{P[A > B] - P[B > A]\}$. Then $A \succ B$ iff $Q(A, B) > 0$.

Lemma 1. *For $n \ge 1$, if Z is the standard die and A is any die, then $Q(Z, A) = 0$.*

Proof. Let $a(i)$ be the number of faces of A with value i. From our assumptions, $\sum_{i=1}^{n} a(i) = n$ and $\sum_{i=1}^{n} ia(i) = n(n+1)/2$. Thus

$$Q(Z, A) = \sum_{i=1}^{n}(n-i)a(i) - \sum_{i=1}^{n} a(i)(i-1) = (n+1)\sum_{i=1}^{n} a(i) - 2\sum_{i=1}^{n} ia(i) = 0,$$

completing the proof. □

The case $n = 4$ begins to show some structure. The five distinct dice are, listing faces, $S_1 = (4, 4, 1, 1)$, $Z = (4, 3, 2, 1)$, $A_1 = (4, 2, 2, 2)$, $A_2 = (3, 3, 3, 1)$, and $S_2 = (3, 3, 2, 2)$. We have listed them in this order to suggest a systematic procedure for their construction. The dice S_1 and S_2, as well as Z, satisfy the following:

Definition 1. *A die S is* symmetric *if, for each $i = 1, 2, \ldots, n$, we have $s(i) = s(n+1-i)$, where $s(i)$ is the number of faces of value i. If a die is not symmetric we call it* asymmetric.

Since $Q(X, Y) = -Q(Y, X)$ the matrix is always antisymmetric. Table 1 displays the Q matrix for the four nonstandard dice in the case $n = 4$. From the matrix, we observe $A_1 \succ S_1 \succ A_2 \succ S_2 \succ A_1$, a nontransitive chain of maximum length. Notice also that $Q(X, Y) = 0$ when X and Y are both symmetric, reminiscent of the fact that $Q(Z, A) = 0$ for all A. This turns out to be generally true, putting a significant restriction on the dominance relationship.

Table 1. The Q matrix for the four nonstandard dice in the case $n = 4$.

		A_1	A_2	S_1	S_2
(4, 2, 2, 2)	A_1	0	−2	2	−2
(3, 3, 3, 1)	A_2	2	0	−2	2
(4, 4, 1, 1)	S_1	−2	2	0	0
(3, 3, 2, 2)	S_2	2	−2	0	0

Lemma 2. *For $n \ge 1$, if S and T are symmetric dice, then $Q(S, T) = 0$.*

Proof. The case $n = 1$ is trivial. For $n \geq 2$, let Q' be the $n \times n$ matrix whose (i,j)th entry is $s(i)t(j)$, where $s(i)$ is the number of faces of S with value i, and $t(j)$ is defined similarly. Then $Q(S,T)$ is equal to the sum of the entries of Q' below the diagonal minus the sum of the entries of Q' above the diagonal. But the (i,j)th entry of Q' below the diagonal is equal to the $(n+1-i, n+1-j)$th entry of Q' above the diagonal, hence the two sums are equal and $Q(S,T) = 0$. □

One might wonder whether $Q(A,B) = 0$ and $Q(B,C) = 0$ implies $Q(A,C) = 0$, as it does in the case $n = 4$. The case $n = 5$ shows this is emphatically not true. In this case, there are 12 distinct admissible dice, six symmetric and six asymmetric. These can be ordered so that $Q(A_i, A_{i+1}) = 0$, $i = 1, 2, \ldots, 11$. Hence any X, Y are joined by a chain such that $Q = 0$ for adjacent elements of the chain. Nonetheless, there are many pairs (A, B) such that $Q(A,B) \neq 0$, as we shall show when we prove the main theorem. How many distinct admissible dice are there for each n? The next lemma answers this for symmetric admissible dice.

Lemma 3. *For $n \geq 1$ there are*

$$S_n = \binom{n-1}{\lfloor (n-1)/2 \rfloor}$$

symmetric admissible dice.[3]

Proof. With the understanding that $0! = 1$ the case $n = 1$ is trivial.

(a) Let $n = 2m \geq 2$ be a positive even integer. There is a one-to-one mapping from the set of dice whose faces are chosen from the first n integers but are not necessarily admissible onto the distinct arrangements of n balls in n ordered cells $\{1, 2, \ldots, n\}$. To obtain this mapping, let the n possible values of a die correspond in order to the n cells and let the number of faces of value k correspond to the number of balls placed in the kth cell. The number of ways to place r balls in n cells is the binomial coefficient $\binom{n+r-1}{r}$. See e.g. Feller (1968). It follows from the definition that the arrangement for symmetric dice is determined once $n/2 = m$ balls have been placed in the first m cells. Setting $n = r = m$ in $\binom{n+r-1}{r}$ shows this can be done in $\binom{2m-1}{m} = \binom{2m-1}{m-1}$ ways.

(b) If $n = 2m+1 \geq 3$ is odd, there must be an odd number $k = 1, 3, 5, \ldots, 2m+1$ of faces with the median value of $m+1$. This leaves $p = 0, 1, 2, \ldots, m$ faces to be assigned values from 1 to m, and symmetry then determines the m remaining values that occur between $m+2$ and $2m+1$. Thus, there are a total of $1 + \sum_{p=1}^{m} \binom{m+p-1}{p}$ symmetric dice, which can be written $\sum_{p=0}^{m} \binom{m+p-1}{p}$. It can easily be shown (e.g., Feller 1968, Chapter II, eq. (12.8)) that $\sum_{p=0}^{m-1} \binom{m+p-1}{p} = \binom{2m-1}{m} \equiv S_{2m}$ from which $S_{2m+1} = 2S_{2m} = \binom{2m}{m}$. The result follows by combining cases (a) and (b). □

[3] The symbol $\lfloor x \rfloor$ stands for the greatest integer less than or equal to x (the "floor" of x).

Using the well known asymptotic estimate based on Stirling's formula, we have

$$S_{2m} \sim \frac{2^{2m-1}}{\sqrt{\pi m}} \quad \text{and} \quad S_{2m+1} \sim \frac{2^{2m}}{\sqrt{\pi m}}.$$

For each n the total number N_n of admissible dice equals the number of partitions of the positive integer $n(n+1)/2$ into a sum of positive integers, subject to the constraints that the partition consists of precisely n parts, and that each part is less than or equal to n. The unconstrained partition problem and many constrained cases have been studied extensively. Riordan (1958) gives the enumerating generating function for partitions with exactly k parts and a maximum part less than or equal to j as

$$p_j(t,k) = t^k \frac{(1-t^j)(1-t^{j+1}) \cdots (1-t^{j+k-1})}{(1-t)(1-t^2) \cdots (1-t^k)}.$$

Putting $j = k = n$ and deleting the factor $(1-t^n)$ from both numerator and denominator, we have the enumerating generating function for partitions with exactly n parts (the n faces of our die) and maximum part less than or equal to n (the value of each face is less than or equal to n),

$$p_n(t,n) = t^n \frac{(1-t^{n+1})(1-t^{n+2}) \cdots (1-t^{2n-1})}{(1-t)(1-t^2) \cdots (1-t^{n-1})}.$$

The coefficient of $t^{n(n+1)/2}$ counts all such partitions of $n(n+1)/2$, hence is N_n. For reference, in Table 2 we give a few computed values for N_n along with values for the number S_n of symmetric dice derived from Lemma 3. Note that when $p = 2^k - 1$ and k is a positive integer, then it is easy to show that the binomial coefficients $\binom{p}{j}$, $j = 0, 1, \ldots, p$, are all odd. Hence when $n = 2^k$, $S_n = \binom{2m-1}{m}$ is odd, as Table 2 shows for S_4 and S_8. The numbers N_n have appeared in the mathematical literature. The *On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences` gives the sequence $1, 2, 5, 12, 32, 94, 289, \ldots$ as the sequence whose nth term is the "number of partitions of the nth triangular number involving only the numbers $1, 2, \ldots, n$ and with exactly n terms," and also as "balancing weights on the integer line: the number of solutions for n weights in distinct integer positions on $[-n, n]$ with a pivot at 0."

We are now ready to prove Theorem 2, which has Theorem 1 as a corollary.

Theorem 2. *Assume $n \geq 4$. (A) For each asymmetric A there exist symmetric S_j and S_k such that $S_k \succ A \succ S_j$. (B) For each nonstandard symmetric S there exist asymmetric A_j and A_k such that $A_k \succ S \succ A_j$. Thus every nonstandard die both dominates and is dominated by another die.*

Proof. If X is any die, we also use X to represent a random variable on n equally likely points having values equal to the values of the faces of X. We

Table 2. The number of admissible dice N_n and the number of symmetric admissible dice S_n.

n	m	S_n	N_n
2	1	1	1
3	1	2	2
4	2	3	5
5	2	6	12
6	3	10	32
7	3	20	94
8	4	35	289
9	4	70	910
10	5	126	2934

may describe X by listing its faces, typically in either monotonic nondecreasing or monotonic nonincreasing order. The standard die in this description would be $Z_n = (1, 2, \ldots, n)$, where we use round parentheses in this representation. It is also useful to describe a die, as we have done, by listing the number of faces $x(i)$ for each value i, in order of increasing value i. Then Z_n is the n-tuple $\{1, 1, \ldots, 1\}$, where we use curly brackets to distinguish this representation from the previous one.

Intuitively, imagine the points $i = 1, 2, \ldots, n$ on the real line with masses $x(i)$ at each i. Then $x(i) = nP[X = i]$ and by assumption $\sum_{i=1}^{n} x(i) = n$ and $\sum_{i=1}^{n} ix(i) = n(n+1)/2$ for admissible dice. So the center of mass is at the midpoint $(n+1)/2$.

Definition 2. *For any die X, the reflection X^* of X is the die such that $x^*(i) = x(n+1-i)$ for $i = 1, 2, \ldots, n$.*

Thus X is symmetric iff $X^* = X$. The reflection transformation can be thought of as a physical reflection of the masses $\{x(1), x(2), \ldots, x(n)\}$ through the center of mass. The next lemma, a collection of facts that we shall use in the proofs, follows easily.

Lemma 4. *For any X, Y, we have $X^{**} = X$, $Q(X,Y) = -Q(Y,X)$, and $Q(X,Y) = -Q(X^*,Y^*)$. Equivalently $X \succ Y \Leftrightarrow Y^* \succ X^*$; $Q(X,Y) = 0 \Leftrightarrow Q(Y,X) = 0 \Leftrightarrow Q(X^*,Y^*) = 0$.*

We return to the proof of Theorem 2, part (A), case of even $n = 2m \geq 4$: Define special symmetric dice S_k, $k = 1, 2, \ldots, m$, by $s_k(k) = s_k(n+1-k) = m$, $s_k(i) = 0$ otherwise. Given any A (asymmetric), it suffices to prove there is an S_k such that $S_k \succ A$, because this means there is an S_j such that $S_j \succ A^*$, and hence $A = A^{**} \succ S_j^* = S_j$.

If $P[S_k > A] - P[A > S_k] > 0$ for some k, we are done, so assume not. Then for $k = 1, 2, \ldots, n/2 = m$ we have $P[S_k > A] \leq P[A > S_k]$, i.e.,

$$\frac{1}{2n}\left\{\sum_{i=1}^{n-k} a(i) + \sum_{i=1}^{k-1} a(i)\right\} \leq \frac{1}{2n}\left\{\sum_{i=n+2-k}^{n} a(i) + \sum_{i=k+1}^{n} a(i)\right\}$$

which leads to a series of inequalities, corresponding to $k = 1, 2, \ldots, m$:

$$\begin{aligned}
a(1) &\leq a(n)\\
2a(1) + a(2) &\leq a(n-1) + 2a(n)\\
2a(1) + 2a(2) + a(3) &\leq a(n-2) + 2a(n-1) + 2a(n)\\
&\vdots\\
2a(1) + \cdots + 2a(m-2) + a(m-1) &\leq a(m+2) + 2a(m+3) + \cdots + 2a(n)\\
2a(1) + \cdots + 2a(m-1) + a(m) &\leq a(m+1) + 2a(m+2) + \cdots + 2a(n)
\end{aligned}$$

If all these inequalities are equalities, then $a(i) = a(n+1-i)$, $i = 1, 2, \ldots, m$, and A is symmetric, contrary to assumption. Hence, at least one is a strict inequality. Thus, summing yields the inequality

$$(n-1)a(1) + (n-3)a(2) + \cdots + a(m) < a(m+1) + 3a(m+2) + \cdots + (n-1)a(n).$$

Putting all the terms on the right side and adding $n(n+1) = (n+1)\sum_{i=1}^{n} a(i)$ yields

$$(n+1)n < 2\sum_{i=1}^{n} ia(i) = (n+1)n,$$

a contradiction that proves this part.

Part (A), case of odd $n = 2m+1$: The proof proceeds as in the even case with the same set of inequalities, to which we adjoin the case $k = m+1$:

$$a(1) + a(2) + \cdots + a(m) \leq a(m+2) + a(m+3) + \cdots + a(n),$$

which is obtained by defining S_{m+1} as the symmetric die such that $s_{m+1}(m+1) = n$ and $s_{m+1}(i) = 0$ otherwise. The calculation of $Q(S_{m+1}, A)$ then yields the $k = m+1$ inequality above. Adding the inequalities for $k = 1, 2, \ldots, m+1$ and assuming the result is a strict inequality leads to a contradiction as in the even case, proving this part.

Definition 3. *A* partition *of the die X of dimension n with the row vector $X = \{x(1), x(2), \ldots, x(n)\}$ is a collection of row vectors also of length n with nonnegative integer coordinates,*

$$X_j = \{x_j(1), x_j(2), \ldots, x_j(n)\}, \quad j = 1, 2, \ldots, k,$$

such that $X = \sum_{j=1}^{k} X_j$. Call such each X_j a subdie. *A subdie X_j is* admissible *if*

$$\sum_{i=1}^{n} ix_j(i) = \frac{n+1}{2}\sum_{i=1}^{n} x_j(i),$$

i.e., if the "center of mass" of the subdie is the same as for an admissible die of the same dimension. A partition is admissible *if each subdie is admissible.*

Every X has a partition $X = X_A + X_S$ into X_A asymmetric and X_S symmetric, where X_S is the maximum symmetric subdie. To construct X_S, simply remove symmetric pairs of masses, one pair at a time, from X until no more remain. If $n = 2m+1$ is odd, also remove the masses at $m+1$. The residual is X_A, and $X - X_A = X_S$. The subdie X_A is "purely asymmetric," i.e., it has the property that $x_A(i)x_A(n+1-i) = 0$, $i = 1, 2, \ldots, n$. We shall call X_A the asymmetric part of X and X_S the symmetric part.

Lemma 5. *Suppose $X = X_A + X_S$, where X_A is purely asymmetric and X_S is symmetric. Then X_A is admissible iff X is admissible. X is symmetric iff $X_A = 0$. If $(X_1, X_2, \ldots, X_k)$ is a partition of X and $(Y_1, \ldots, Y_p)$ is a partition of Y, then $Q(X,Y) = \sum_{i=1}^{k}\sum_{j=1}^{p} Q(X_i, Y_j)$. In particular,*

$$Q(X_A + X_S, Y_A + Y_S) = Q(X_A, Y_A) + Q(X_A, Y_S) + Q(X_S, Y_A),$$

and if Y is symmetric, then $Q(X_A + X_S, Y) = Q(X_A, Y)$.

Proof. The various statements in the lemma follow directly from the definitions. The last fact allows us to simplify the proof of part (B) of Theorem 2 by using simple purely asymmetric subdice to dominate any symmetric S. □

Part (B), case of $n = 2m \geq 4$: Choose an arbitrary fixed symmetric $S \neq Z_n$. If $s(k-1) = s(k+1)$ for all $k = 2, 3, \ldots, n-1$, then $s(1) = s(3) = \cdots = s(n-1) = x$, while $s(2) = s(4) = \cdots = s(2n) = y$. But $s(1) = s(2n)$ from the symmetry of S, hence $x = y$ and this common value must be 1 since $\sum_{i=1}^{n} s(i) = n$, hence $S = Z_n$, a contradiction. Hence we must have $s(k-1) \neq s(k+1)$ for some k. Without loss of generality we may assume $k \leq m$ since S is symmetric. If $k < m$, define X_A by $x_A(k-1) = x_A(k+1) = 1$, $x_A(n+1-k) = 2$, $x_A(i) = 0$ otherwise. If $k = m$, define X_A by $x_A(m-1) = 1$, $x_A(m+1) = 3$, $x_A(i) = 0$ otherwise. Note that X_A is admissible. A calculation shows that $Q(X_A, S) = s(k+1) - s(k-1) \neq 0$, hence either $X_A \succ S$ or $X_A{}^* \succ S$.

Part (B), case of $n = 2m+1 \geq 5$: For some k, $s(k-1) \neq s(k+1)$. To prove this, suppose instead that $s(k-1) = s(k+1)$ for every k such that $2 \leq k \leq n-1$. Then since n is odd and S is symmetric, $s(m+1) = x > 0$ where x is odd. Hence $s(i) = x$ for all odd i. For any admissible die other where x is odd. Hence $s(i) = x$ for all odd i. For any admissible die other than Z_n there must be at least one index i such that $x(i) = 0$ (otherwise the sum of the masses would exceed n.) In fact it follows at once from the fact that the die is admissible that there must be at least two such points. If $x(i) = 0$, i must be even, hence $x(i) = 0$ for all even i. Then the total mass, which is $2m+1$, must equal $(m+1)x$, where x is an odd integer. But then $x = (2m+1)/(m+1)$ which is not an integer. We have a contradiction, and therefore there is a k such that $s(k-1) \neq s(k+1)$.

From the symmetry of S, $k \neq m+1$ and we can choose $k < m+1$. As before, define X_A by $x_A(k-1) = x_A(k+1) = 1$, $x_A(n+1-k) = 2$, $x_A(i) = 0$ otherwise. As before, a calculation shows that $Q(X_A, S) = s(k+1) - s(k-1) \neq 0$.

One way to do the calculation is via the following easily proven lemma.

Lemma 6. *If D is any subdie, not necessarily admissible, with reflection D^* and S is symmetric, then $Q(D, S) = -Q(D^*, S)$.*

Now decompose the X_A of the proof of the theorem into $X_A = D_1 + D$ where D_1 is defined by $d_1(k-1) = d_1(k+1) = 1$, $d_1(i) = 0$ otherwise, an D_2 is defined by $d_2(n+1-k) = 2$, $d_2(i) = 0$ otherwise. Then

$$Q(X_A, S) = Q(D_1, S) + Q(D_2, S) = Q(D_1, S) - Q(D_2{}^*, S) = s(k+1) - s(k-1)$$

where the calculation for the last equality is direct. Note that this metho unifies the $k < m$ and $k = m$ cases for the proof of part (B), case of $n = 2m \geq 4$. This completes the proof of Theorem 2. □

To see why the special S_k are sufficient for determining whether or not given A, there is an S such that $Q(S, A) \neq 0$, we note the following lemma.

Lemma 7. *For all $n \geq 4$, with S_k as in Theorem 2, part (A), given any fixe asymmetric die A we have $\max_k Q(S_k, A) = \max_S Q(S, A)$.*

Proof. We give the proof for even $n = 2m \geq 4$. The proof for odd n is similar a bit more involved, and is omitted. Suppose $n = 2m \geq 4$. Define symmetri subdice X_k for $k = 1, 2, \ldots, m$ by $x_k(k) = x_k(n+1-k) = 1$, $x_k(i) =$ (otherwise. Any symmetric S can be written as $S = \sum_{k=1}^{m} c_k X_k$ where the c_l are nonnegative integers and $2\sum_{k=1}^{m} c_k = n$. Then

$$Q(S, A) = \sum_{k=1}^{m} c_k Q(X_k, A) \leq \sum_{k=1}^{m} c_k M = mM,$$

where $M = \max_k Q(X_k, A)$. Choose $k = p$ such that $M = Q(X_p, A)$. Ther $Q(S, A) \leq mQ(X_p, A) = Q(S_p, A)$. □

3 A nontransitive-dice game

Suppose for a given $n \geq 4$ that we have a bucket containing all the admissible dice except Z_n. Player I chooses a die X which is shown to Player II. Player II then chooses a die Y. Both dice are rolled, and if one player's die shows a higher number than the other player's, the latter pays the former one unit. The expected payoff to Player II is $E = Q(Y, X)/n^2$. Theorem 2 shows that Player II can always choose Y so that $E > 0$. If both players choose optimally, we have the value of the game to Player II as $V_{II} = \min_X \max_Y Q(Y, X)/n^2$. Examination of the matrix Q shows that for $n = 4$ $V_{II} = 1/8$, for $n = 5$ $V_{II} = 1/25$, and for $n = 6$ $V_{II} = 1/36$.

For those interested in playing the game, we find that for $n = 6$ there are no nontransitive chains $A_1 \succ A_2 \succ \cdots \succ A_{j+1} = A_1$ with $Q(A_i, A_{i+1}) > 6$ for $i = 1, 2, \ldots, j$. There are precisely two with $Q(A_i, A_{i+1}) \geq 6$. They are

$S_2 \succ A \succ A^* \succ S_2$ and $S_2 \succ A \succ S_1 \succ A^* \succ S_2$, where $S_1 = (3,3,3,4,4,4)$, $S_2 = (2,2,2,5,5,5)$, and $A = (1,4,4,4,4,4)$. Note that $\max_{X,Y} Q(X,Y) = Q(A, A^*) = 14$.

Before studying V_{II} further, we investigate how badly Player I can play if Player II plays optimally; i.e., we study $\max_X \max_Y Q(Y,X)$.

Lemma 8. *(a) If $n = 2m \geq 4$ and Player I chooses A_n defined by $a_n(m) = n-1$, $a_n(n) = 1$, and $a_n(i) = 0$ otherwise, then the value to Player II is at least $1 - 4/n + 2/n^2$.*

(b) If $n = 2m+1 \geq 5$ and Player I chooses S_n defined by $s_{m+1}(m+1) = n$ and 0 otherwise, then the value to Player II is at least $1 - 4/n$.

Proof. The idea is to pack the masses of Player I's die as much as possible on one point and also as close to the midpoint as possible, then construct a die for Player II with as much mass as possible one step above.

Proof of (a): If $m \geq 2$, define $Y = A_n{}^*$. For A_n, verify that $\sum_{i=1}^n a(i) = n$, $\sum_{i=1}^n ia(i) = n(n+1)/2$. We have $Q(A_n{}^*, A_n) = n^2 - 4n + 2$.

Note that for $n = 4$ we get $Q(A_n{}^*, A_n) = 2$, which is $\max_{X,Y} Q(X,Y)$. For $n = 6$ we get $Q(A_n{}^*, A_n) = 14$, which also is $\max_{X,Y} Q(X,Y)$. For $n = 8$, $Q(A_n{}^*, A_n) = 34$.

Proof of (b): Define A by $a(1) = a(2) = 1$, $a(m+2) = n-2$, and 0 otherwise. Note that since $m+1 \geq 3$, the masses at 1 and 2 are "below" the masses at $m+1$. Then observe that $\sum_{i=1}^n a(i) = n$, $\sum_{i=1}^n ia(i) = n(n+1)/2$, and $Q(A, S_n) = n(n-4)$. For $n = 5$, $Q(A, S_n) = 5 = \max_{X,Y} Q(X,Y)$. □

Note that changing A slightly to B, defined by $b(1) = 2$, $b(m+2) = n-3$, $b(m+3) = 1$, and 0 otherwise, works equally well.

Now we look at how well Player I can do. Define $\lfloor x \rfloor$ as the integer part of x, and $\mathrm{fr}(x)$ as the "fractional part" of x, i.e. $\mathrm{fr}(x) = x - \lfloor x \rfloor$.

Theorem 3. *Define the die X_0 by $x_0(1) = x_0(n) = 0$, $x_0(2) = x_0(n-1) = 2$, $x_0(i) = 1$ otherwise. Then for $n \geq 6$, $n^2 V_{II} \leq \max_Y Q(Y, X_0) = \lfloor \frac{n(n-5)}{2(n-4)} \rfloor$ and a maximizing Y is given by*

(i) $y(3) = a = \frac{n(n-3)}{2(n-4)}$, $y(n-1) = b = \frac{n(n-5)}{2(n-4)}$, and 0 otherwise when a is an integer, and

(ii) $y(3) = \lfloor a \rfloor$, $y(n-1) = \lfloor b \rfloor$, $y(n-1-k) = 1$, where $0 < k < n-4$ (recall that $n \geq 6$), and $k = (n-4)\,\mathrm{fr}(a)$ and $y(i) = 0$ otherwise, when a is not an integer.

An upper bound for V_{II} is therefore $V_{II} \leq \frac{n-5}{2n(n-4)} < \frac{1}{2n}$.

Proof. Since $n^2 V_{II} = \min_X \max_Y Q(Y,X) \leq \max_Y Q(Y, X_0)$, the latter yields an upper bound for the value of the game to Player II. The motivation for the choice of X_0 is that if Player I could choose Z_n then V_{II} would equal 0, and X_0 is "as close as possible" to Z_n. We write $X_0 = Z_n - X_1 + X_2$, where X_1 and X_2 are as in Lemma 7. Choosing X_1 and X_2 from the set of X_k appears to give the "best" constraints on the $y(i)$.

A calculation yields $Q(Y, X_0) = y(n) + y(n-1) - y(2) - y(1)$ and Player II wishes to maximize this. One would expect this to occur when $y(n-1)$ is maximized subject to $y(1) = y(2) = y(n) = 0$. We shall prove this later. Assuming this for now, imagine that $y(3) = a$, $y(n-1) = b$, where a and b are positive real numbers, and $y(i) = 0$ otherwise. Then a and b must satisfy

$$a + b = n,$$
$$3a + (n-1)b = n(n+1)/2,$$

the solution of which is $a = \frac{n(n-3)}{2(n-4)}$ and $b = \frac{n(n-5)}{2(n-4)}$. If a and b are integers, this establishes part (i). If a and b are not both integers, then since $n(n-3)/2$ is an integer, a has the form $a = \lfloor a \rfloor + q$ where $0 < q < 1$ and $q = k/(n-4)$ for some integer k such that $0 < k < n-4$. Since $3\,\mathrm{fr}(a) + (n-1)\,\mathrm{fr}(b) = 3k/(n-4) + (n-1)(1 - k/(n-4)) = n-1-k$, (ii) follows.

It remains to prove that the chosen Y maximizes $Q(Y, X_0)$. We indicate the method, which can be formalized. Start with any A chosen by Player II. We show $Q(Y, X_0) \geq Q(A, X_0)$.

Step 1: If $a(n) \neq 0$, shift one unit of mass from n to $n-1$ and simultaneously shift one unit of mass one position to the right from the least k such that $a(k) \neq 0$. This keeps A admissible. Iterate this procedure (Step 1) until $a(n) = 0$. Call this result A_1. Clearly, $Q(A_1, X_0) \geq Q(A, X_0)$.

Step 2: If $a(1) \neq 0$ or $a(2) \neq 0$, and some $a(k) \neq 0$ for $3 < k < n-1$, then shift one unit of mass from the greatest such k to $k-1$ and simultaneously shift one unit of mass from the lesser of 1 or 2 to one index higher. Iterate this procedure (Step 2) until either (a): $a(k) = 0$ for all k such that $3 < k < n-1$ or (b): $a(1) = a(2) = 0$. Call the result A_2. Clearly $Q(A_2, X_0) \geq Q(A_1, X_0)$.

Step 3(i): For A_2 we have $a_2(k) = 0$ when $3 < k < n-1$ and $a_2(n) = 0$. Hence $a_2(1)$ or $a_2(2) \neq 0$, and further $a_2(n-1) \neq 0$ as well because of the condition $\sum_{i=1}^n ia(i) = n(n+1)/2$. So long as $a_2(1)$ or $a_2(2) \neq 0$, shift one of the masses up one step and simultaneously shift down one step one mass from the least $k > 3$ such that $a_2(k) \neq 0$, except when the lower shift is from $a_2(1)$ and the upper shift is from $a_2(n-1)$. In this latter case, shift each two steps. This last is to avoid increasing $Q(A, X_0)$. Iterate until $a_2(1) = a_2(2) = 0$. The result is A_3, where $a_3(3) \neq 0$, $a_3(n-1) \neq 0$, $a_3(k) \neq 0$ for at most one k such that $3 < k < n-1$, and $a_3(i) = 0$ otherwise. Clearly, A_3 satisfies $Q(A_3, X_0) \geq Q(A_2, X_0)$ and also the conditions of the theorem.

Step 3(ii): Here A_2 satisfies $a(1) = a(2) = a(n) = 0$. Now we want to push up the masses between $k = 3$ and $n-2$ with the largest k towards $k = n-1$ and those with the smallest k towards $k = 3$. If $\sum_{i=4}^{n-2} a(i) \leq 1$, we are done. If instead there are two or more masses in the range $3 < k < n-1$, move any such mass with greatest k one step up, and balance by moving any such mass with least k one step down. Iterate until either all masses are at $k = 3$ and $k = n-1$ or all but one is, and it necessarily satisfies $3 < k < n-1$. Call the result A_3. Again, $Q(A_3, X_0) \geq Q(A_2, X_0)$, and A_3 is the Y of the theorem. Thus the Y of the theorem maximizes $Q(Y, X_0)$. This completes the proof of Theorem 3. □

A computer program verifies that for $n = 6$ and $n = 8$, the X_0 and Y of the theorem are optimal strategies and, along with their "dual" strategy pairs, are the only optimal strategies. The value V_{II} for $n = 8$ is 3/64. The "dual" strategy pairs to $X_0 = Z_n - X_1 + X_2$ and Y are defined to be ${X_0}^D = Z_n + X_1 - X_2$ and Y^*, respectively. For $n = 7$ the computer verifies that $V_{II} = 2/49$ and that the X_0 and Y of the theorem are optimal strategies. The remaining optimal strategy pairs consist of X_0 and the two "obvious" perturbations of Y: $Y_1 = (1333666)$ and $Y_2 = (3333367)$, along with their reflected strategy pairs. The reflection of $X = (x(1), x(2), \ldots, x(n))$ is $X^* = (n+1-x(1), n+1-x(2), \ldots, n+1-x(n))$, and the reflected pair for (Y, X_0) is $(Y^*, {X_0}^D)$.

4 Cycles, chains, and nontransitivity in money manager performance comparisons

We saw in the discussion of $n = 4$ which followed Lemma 1 that the four dice other than Z formed a closed loop with no repeats, which we defined as a cycle. It follows at once from the Q matrix for $n = 4$ that this is the only cycle of (maximum) length 4: $A_1 \succ S_1 \succ A_2 \succ S_2 \succ A_1$. (The standard die Z is omitted throughout this discussion unless explicitly introduced.) There is also exactly one cycle of length 3, namely $A_1 \succ S_1 \succ A_2 \succ A_1$. One way to see this is to define a new matrix R by $R(i,j) = 1$ if $Q(i,j) > 0$ and $R(i,j) = 0$ otherwise. Of course, from the antisymmetry of Q we have $R(i,j)R(j,i) = 0$ for all i, j. In Table 3 we give the R matrix corresponding to the Q matrix in Table 1.

Table 3. The R matrix for the four nonstandard dice in the case $n = 4$.

		A_1	A_2	S_1	S_2
$(4,2,2,2)$	A_1	0	0	1	0
$(3,3,3,1)$	A_2	1	0	0	1
$(4,4,1,1)$	S_1	0	1	0	0
$(3,3,2,2)$	S_2	1	0	0	0

To see how to use R to find all the cycles, define a principal submatrix of R as one constructed by choosing a subset of the rows of R and exactly the same subset of columns. A permutation matrix is said to be "contained" in R if it is a permutation matrix whose nonzero elements are all nonzero elements of some principal submatrix of R. The next result is easily shown:

Lemma 9. *There is a one-to-one correspondence between permutation matrices contained in R and cycles. Given either the other can be directly computed.*

Cycles are interesting for a number of reasons. For instance, any cycle yields a set of dice that can be used to play the game of Section 3. As another instance we shall use a cycle of nontransitive dice to construct a hypothetical example that shows the nontransitivity of a criterion that might be used to compare the performance of mutual fund managers.

This leads us to look for "large" cycles, and in particular cycles of maximum length for a given n. In preparation for our example, we find all cycles of maximum length for $n = 5$. There are $N_5 = 12$ admissible dice, listed in Table 4. In Table 5 we display R for the case $n = 5$, derived from a computer calculation of Q, and presented with a permutation of the rows and columns that will lead to a certain order and simplicity. Note that Z has been omitted from the matrix.

Table 4. The 12 admissible dice for $n = 5$.

die no.	name	faces	die no.	name	faces
1	s_1	33333	7	s_4	53331
2	s_2	43332	8	f_1	54222
3	s_3	44322	9	Z	54321
4	p	44331	10	p_2	54411
5	p_1	44421	11	f_2	55221
6	f	53322	12	c	55311

Note that the last 5 rows and 5 columns correspond to the five symmetric dice, producing a 5×5 block of zeros in the lower right-hand corner. We also have examples (not listed here) for which $Q(A, B) = 0$ and both A and B are asymmetric dice, a situation that did not occur for $n = 4$. There are no cases, when $n = 4$ or $n = 5$, where $Q(A, S) = 0$ with A asymmetric and S symmetric. This does occur for $n = 6$.

Now suppose we have a cycle. If $A \succ B$, call A a *predecessor* of B and B a *follower* of A. The four s_i have the same set of three predecessors, $\{p_1, p_2, p\}$ and no others, so at least one s_i cannot appear in a cycle. Thus any cycle has length 10 or less. Suppose there is a cycle of length 10. Then it must contain three sequences of the form (p or p_i, s_j, f or f_k) together with the symmetric die c. Since the three ps must precede ss the 1s in rows 1–3, columns 1–6 play no role and can be set to 0. Similarly, since fs must follow ss, the 1s in rows 1–6, columns 4–6 play no role and can be set to 0.

Table 5. The 11×11 R matrix for the case $n = 5$.

die no.	name	p_1	p_2	p	f_1	f_2	f	c	s_1	s_2	s_3	s_4
5	p_1	0	0	1	0	0	1	0	1	1	1	1
10	p_2	1	0	1	0	0	0	0	1	1	1	1
4	p	0	0	0	1	0	1	0	1	1	1	1
8	f_1	1	1	0	0	1	0	1	0	0	0	0
11	f_2	1	1	0	0	0	0	1	0	0	0	0
6	f	0	0	0	1	1	0	1	0	0	0	0
12	c	1	1	1	0	0	0	0	0	0	0	0
1	s_1	0	0	0	1	1	1	0	0	0	0	0
2	s_2	0	0	0	1	1	1	0	0	0	0	0
3	s_3	0	0	0	1	1	1	0	0	0	0	0
7	s_4	0	0	0	1	1	1	0	0	0	0	0

Observing Table 5 now, f must precede c (as the only remaining nonzero entry in the f row is in column c.) Note also that either f_1 precedes p_1 and f_2 precedes p_2 or f_1 precedes p_2 and f_2 precedes p_1. Then c must precede p. Thus, there are cycles of length 10, and they are all of the form (p, s_i, f_j), (p_k, s_ℓ, f_m), (p_n, s_o, f), (c).

The relationship $\succ$ determines a direction on the cycle and c can be thought of an origin of coordinates. Thinking of each cycle as a ring of 10 labeled beads, we can count the number of distinct cycles. The three s's can be arranged in $4 \times 3 \times 2 = 24$ distinct ways. The two f_is and the two p_is can each be arranged in two distinct ways, for a total of 96 distinct cycles.

If we define a chain as a sequence $A_1 \succ A_2 \succ \cdots \succ A_k$, with k as its length, there is a mapping under which each cycle produces three chains of length 11, distinct from all the others. To construct these chains, start with a 10-cycle and sever it between one of the ps and the s that follows it. Then add the unused fourth s as the follower of the final p. This cut can be done in three distinct ways (count from c). Note that c will then appear either in place 3, 6, or 9 of the resulting 11-chain, and that the chain starts and ends with an s. So, there are 288 distinct 11-chains. A computer program confirmed the analysis.

Example 1. Nontransitive dice, the performance of money managers, and match-play golf. Our results for cycles for $n = 5$ yield the following financial paradox: Consider 10 money managers, $i = 0, 1, \ldots, 9$, whose returns R_{it} in period t, $t = 1, 2, \ldots, T$, equal $I_t + e_{it}$, where I_t is the return on an "index" and e_{it} is a random error. In the real world, I_t might be the return on the S&P500 index, the managers might be running funds of large capitalization stocks, and e_{it} is the deviation in performance from that of the index for manager i in period t. If the managers have no skill, which appears to be the predominant state of affairs, and which we assume, then the expectation

$E[e_{it}] = 0$ for all i and t. Fees and trading costs matter but, if we assume they are the same for all 10 managers, then all returns are shifted by the same constant, with no effect on the comparative results.

Next, choose a 10-cycle with distinct dice $A_0 \succ A_1 \succ \cdots \succ A_9 \succ A_0$. For each i, let X_i be the random variable corresponding to A_i and let $\{X_{it} : t = 1, \ldots, T\}$ be independent and identically distributed with the same distribution as X_i. Now define $Y_{it} = a(X_{it} - E[X_i]) = e_{it}$, where a is an arbitrary nonzero scaling constant. Define $R_{it} = I_t + e_{it}$. For dice A and B, define $E(A, B) = Q(A, B)/n^2 = P[A > B] - P[B > A]$, and for random variables X and Y define $E(X, Y) = P[X > Y] - P[Y > X]$. Then it is easy to see that for any i and j,

$$E(A_i, A_j) = E(X_{it}, X_{jt}) = E(Y_{it}, Y_{jt}) = E(e_{it}, e_{jt}) = E(R_{it}, R_{jt}).$$

In the above argument we used the fact that $E[X_i] = m$, a common mean, which is $5/2$ in this particular example. That's why this example works for our nontransitive dice. Thus we have the startling conclusion that, given any manager i in our example, there is another manager $j = i - 1 \pmod{10}$ who, when their two results are unequal, has a probability greater than $1/2$ of outperforming him in any period. As the number of periods increases, the probability that manager $i-1 \pmod{10}$ outperforms manager i in the majority of nontied trials tends to 1.

The following theorem shows that this example can be extended to cycles of arbitrary length.

Theorem 4. *For every $m \geq 3$ there is a cycle of length exactly m, hence for every $m \geq 2$ there is a chain of length m.*

Proof. Let $m \geq 3$ and odd. Construct a set of admissible dice with $n = 3m$ as follows: For $k = 1, 2, \ldots, m-1$ define die A_k by placing mass $3k$ at coordinate $(3m+1)/2 + m - k$ and place mass $3(m-k)$ at coordinate $(3m+1)/2 - k$. One easily checks that the total mass is $3m$ and that the mean is $(3m+1)/2$. Define die A_m by placing mass $3m$ at $(3m+1)/2$. An easy computation shows that $A_1 \succ A_2 \succ \cdots \succ A_m \succ A_1$. To obtain a chain of even length p, let $m = p + 1$ and carry out the above construction. Then note that $A_m \succ A_2$, and hence we can delete A_1 from the cycle to obtain a cycle of length p. Although cycles are discussed in the extensive literature on nontransitive dice and in voting paradoxes (see especially the very general and definitive work of Saari), because of the special nature of the sets of dice we are considering, Theorem 4 does not appear to follow from previous work. □

Theorem 4 shows that an example like the previous one can be constructed for every $m \geq 3$: For every manager there is another who is better under the criterion "*is expected to be ahead more often.*" (Similar examples can be constructed by the same method from more generally defined types of nontransitive dice; however, this typically requires the means $E[e_{it}] = m_i$ to vary from manager to manager.)

If we consider golf players, where a golfer's score on any given day is given by par + A_i, $i = 0, \ldots, m-1$, and A_i are a cyclic nontransitive set with $A_0 \succ A_1 \succ \cdots \succ A_{m-1} \succ A_0$, then in match play $E(i, i+1) > 0$ for all i (mod m). If the dice are chosen as above the golfers all have the same expected scores (like our money managers). Note that these same examples will apply to tournaments involving players of equal skill in a wide variety of settings.

Thus, the criterion "better than," defined to mean "Manager A is better than Manager B" if A is expected to beat B more often than B beats A, can be nontransitive over a set of managers, even though all have the same expected returns in any given period.

The previous examples describe expected outcomes "before the fact" or *ex ante*. Corresponding examples with actual outcomes, i.e., *ex post*, are simpler and more decisive. For instance, suppose we have $m \geq 3$ money managers and their outcomes for m periods are given by $R_{it} = i + t$ (mod m) for manager i, $i = 0, 1, \ldots, m-1$, in period t, $t = 1, \ldots, m$. Then manager i is beaten by manager $i+1$ (mod m) in all but one period, yet the distribution of actual returns over the m periods is identical for the m managers. Only the order in which their returns occur is different.

How do results for m-sided dice relate to results for n-sided dice? We find that the structure that appears at any level m reappears repeatedly as an "embedded subset" for $n > m$ in various ways, as illustrated by the next theorems.

Let D_n^* be the set of all N_n admissible n-sided dice. We represent each such die d_i, $i = 1, \ldots, N_n$, by listing the faces of the die $(x_1, x_2, \ldots, x_n)$ in nondecreasing order. Thus, $Z_4 = (1, 2, 3, 4)$. Each $(x_1, \ldots, x_n)$ is an n-dimensional vector, so we shall call n the *dimension* of D_n^* and speak of the set of n-dimensional dice.

Definition 4 (First embedding). *For each $k \geq 1$ and each $n \geq 2$, the embedding maps $h_{n,k} : D_n^* \to D_{n+k}^*$ are defined by $h_{n,k}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, n+1, \ldots, n+k)$.*

Theorem 5. *The maps $h_{n,k}$ possess the following properties:*

a. $h_{n,k}$ is 1-1.
b. $h_{n,k}(Z_n) = Z_{n+k}$.
c. $h_{n,k}$ preserves Q: $Q(h_{n,k}(A), h_{n,k}(B)) = Q(A, B)$ for all A, B. Hence $h_{n,k}$ is order-preserving as well: $A \succ B$ if and only if $h_{n,k}(A) \succ h_{n,k}(B)$.

Thus the structure, games, and results that depend only upon property (c) and hold for a given n also hold for a subset of each D_{n+k}^*, for each $k \geq 1$.

Definition 5 (Second embedding). *For each n, the embedding maps $f_n : D_n^* \to D_{n+2}^*$ are defined by*

$$f_n(x_1, x_2, \ldots, x_n) = (1, x_1 + 1, x_2 + 1, \ldots, x_n + 1, n + 2).$$

Theorem 6. *For each $n = 1, 2, \ldots$ the maps f_n each have the following properties:*

a. f_n is 1-1.
b. $f_n(Z_n) = Z_{n+2}$.
c. $f_n(B)$ is symmetric or asymmetric according to whether B is.
d. f_n preserves Q: $Q(f_n(A), f_n(B)) = Q(A, B)$. Hence f_n is order-preserving as well: $A \succ B$ if and only if $f_n(A) \succ f_n(B)$.

Thus the structure, games, and results that depend only on these four properties and that hold for a given n also hold for $n + 2k$, $k = 1, 2, \ldots$.

e. The composition map $f_{n+2k} f_{n+2(k-1)} \cdots f_{n+2} f_n : D_n^* \to D_{n+2k}^*$ is defined and has the same properties.
f. $E(f_n(A), f_n(B)) = \left(\frac{n}{n+2}\right)^2 E(A, B)$.

Properties (a)–(e) are obvious. Property (f) follows from the fact that

$$P[A > B] = Q(A, B)/n^2.$$

Since N_n grows rapidly with n, it is not surprising that there are many embedding maps with similar properties.

We wish to thank both the anonymous referee and the editor for their numerous helpful comments and suggestions. The proof of Lemma 2 is an alternative that was suggested by the referee.

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