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De Moivre, Euler, and Montmort analyzed a predecessor of Faro. We consider the modern game first under the assumption of random shuffling, then with nonrandom shuffling. With random shuffling we find the house edge can be less than 0.0006 percent, but it is at least 0.526 percent if the player is limited to negative expectation bets. Human shuffling is nonrandom and a simple model for it indicates that, in principle, the player can achieve significant positive expectation. The ideas used to apply nonrandom shuffling to Faro also extend to other games. We illustrate with casino Blackjack. An appendix discusses previous work on modern Faro.

1. INTRODUCTION

Faro appeared as early as 1400 [4, 8, 13, 20]. One version [19] was mathematically analyzed by De Moivre [2], Euler [6], and Montmort [12].

We present what we believe is the first correct treatment of modern Faro. (See the appendix for a discussion of the literature.) Section 2 assumes random shuffling and estimates the house edge on that basis. Section 3 observes that real shuffling is nonrandom and uses a simple model of it to indicate that the player can, in principle, achieve significant positive expectation. Though Faro is rarely played now, the methods apply to other games, as we illustrate with casino Blackjack.

The reader interested in nonrandom shuffling may skip directly to Section 3 after reading Subsections 2.1 and 2.5. An expanded version of this article is available from the author.

2. PROBABILITIES AND STRATEGIES WITH RANDOM SHUFFLING

2.1 Rules and Procedures

Pairs of cards are dealt face up from a single 52-card (bridge) deck. The first card of the deck (soda) is inactive. Then cards are dealt in pairs. The first of the pair is the losing card, and the second is the winning card. The losing card is drawn from the box and placed immediately to the side in the *losing pile.* The winning card stays momentarily in the box. After the settlement of this *hand,* or *turn,* bets may be made or withdrawn. Then another pair of cards is drawn. (The first is the winner on the previous turn and is placed on the winning pile.) The process continues until 25 pairs or 50 cards, plus the original soda card, have been played.

If a player bets one unit on a specified rank to win (say Q) then he wins one unit if the next two cards are (X, Q) in that order, where X is any card not of rank Q. He loses his bet if the order is (Q, X), and loses half his bet if (Q, Q) occurs. If (X, X) occurs, he neither wins nor loses.

We limit analysis here to the bet on the rank to win. Other bets are discussed in the references, as are variations to the game. The expanded version of this article extends the methods herein to these bets and to the variations.

2.2 The Bet on a Rank

Suppose the player bets on the rank Q to win when m cards of rank Q and n cards of other ranks X remain. The remaining number of turns is $t = [(m + n)/2]$, where $[x]$ is the integer part of x. Assume the player leaves his bet on until a Q appears and the bet is settled, or for k turns, where $1 \leqq k \leqq t$. He removes the bet after k turns if the outcomes have all been (X, X).

We wish to find $W(m, n; k)$, the probability that the player wins; $L(m, n; k)$, the probability that the player loses; $S(m, n; k)$, the probability that the player loses half his bet; and $N(m, n; k)$, the probability of no resolution. We also wish to know $G(m, n; k)$, the player's expectation per unit bet, and $G_C(m, n; k)$, the player's (conditional) expectation per unit bet, given that (X, X) does not occur on all k turns, i.e., that the bet is resolved.

It can be shown that

$$G = W - L - S/2 = -S/2,$$
$$G_C = G/(1 - N),$$
$$W + L + S + N = 1 \quad \text{so} \quad W + L = 1 - (S + N),$$

and

$$W = L = (1 - S - N)/2.$$

Thus S and N yield all the desired quantities.

$N(m, n; k)$ is the probability that the first $2k$ cards are all X. Therefore $N(m, n; k) = 0$ if and only if $2k > n$. If $2 \leqq 2k \leqq n$,

$$N(m, n; k) = \binom{n}{2k} \Big/ \binom{m+n}{2k} = (n)_{2k}/(m + n)_{2k}$$
$$= (m + n - 2k)_m/(m + n)_m.$$

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If $m = 1$, $S(m, n; k) = 0$. If $m = 2$, there are $n + 2$ cards remaining. A tie can occur only if the two Q cards are in positions (1, 2), (3, 4), $\cdots$, $(2k - 1, 2k)$, respectively. Thus there are k ways to select the location of the two Q cards to terminate a bet on Q in a tie during the first k turns. There are $\binom{n+2}{2}$ equiprobable locations in all for the two Q cards. Thus

$$S(2, n; k) = \binom{n+2}{2}^{-1} k = 2k/(n+2)_2$$

when $1 \leqq k \leqq [n/2] + 1 \equiv t$.

In actual play $m + n$ is always odd, hence n is odd if $m = 2$. If a bet is allowed (or required) to remain until it is resolved or the last turn ends, we have

$$S(2, n; t) = 1/(n + 2).$$

If $m > 2$, there are $\binom{m+n}{m}$ equiprobable ways to select the locations of the m cards of rank Q. If $2j \leqq n + 2$ then $\binom{m+n-2j}{m-2}$ such selections end the wager in a tie on the jth turn. A Q must appear for some $j \leqq [n/2] + 1$ so the wager cannot end in a tie when $2j > n + 2$. Thus,

$$S(m, n; k) = \binom{m+n}{m}^{-1}\left\{\binom{m+n-2}{m-2} + \binom{m+n-4}{m-2} + \cdots + \binom{m+n-2k'}{m-2}\right\} \quad (2.1)$$

where $k' = \min(k, [n/2] + 1)$.

The method of Bernoulli and Montmort (see [19, pp. 89–90]) yields for $m > 2$ the equivalent forms

$$S(m, n; k) = \binom{m+n}{m}^{-1}\left\{\sum_{i=1}^{m-2} 2^{-i}\left[\binom{n+m-i}{m-i} - \binom{n-2k'+m-i}{m-i}\right] + k'2^{2-m}\right\} \quad (2.2)$$

$$S(m, n; k) = \binom{m+n}{m}^{-1}\left\{\sum_{i=1}^{m-2} - (-2)^{-i}\left[\binom{m+n}{m-i} - \binom{m+n-2k'}{m-i}\right] + (-2)^{2-m}k'\right\}. \quad (2.3)$$

When $m = 3$ we have

$$S(3, n; k) = \binom{n+3}{3}^{-1} k'(n + 2 - k')$$

and for a bet left on for $n/2 + 1 = t$ turns,

$$S(3, n; t) = 3(n + 2)/2(n + 1)(n + 3).$$

When $m = 4$ then

$$S(4, n; k) = \tfrac{1}{2}\binom{n+4}{4}^{-1} \cdot \{[(n + 4)_3 - (n + 4 - 2k)_3]/3! - k(n + 3 - k)\}$$

or equivalently

$$\tfrac{1}{2}\binom{n+4}{4}^{-1}\{[(n + 3)_3 - (n + 3 - 2k)_3]/3! + k(n + 2 - k)\}.$$

Now, n is odd in actual play and the wager must terminate for some $j \leqq [n/2] + 1 = (n + 1)/2 = t$. Thus $S(4, n; t) = S(4, n; t + 1) = (2n + 7)/(n + 2)(n + 4)$.

The general formulas become: For $m = 1$: $S = 0$; $N = (n - 2k + 1)/(n + 1)$; $G = G_C = 0$. For $m = 2$:

$$S(2, n; k) = 2k/(n + 2)_2;$$
$$N(2, n; k) = (n - 2k + 2)_2/(n + 2)_2;$$
$$G(2, n; k) = -k/(n + 2)_2;$$
$$G_C(2, n; k) = -\tfrac{1}{2}[2(n - k) + 3].$$

For $m = 3$:

$$S(3, n; k) = \binom{n+3}{3}^{-1} k'(n + 2 - k');$$
$$N(3, n; k) = (n + 3 - 2k')_3/(n + 3)_3;$$
$$G = -\tfrac{1}{2}\binom{n+3}{3}^{-1} k'(n + 2 - k');$$
$$G_C(3, n; k) = -3k'(n + 2 - k')/[(n + 3)_3 - (n + 3 - 2k')_3].$$

For $m \geqq 3$: $S(m, n; k)$ is given by any of forms (2.1), (2.2) or (2.3); $N(m, n; k) = (n)_{2k}/(m + n)_{2k}$; $G = -S/2$; $G_C = -S/2(1 - N)$.

2.3 Numerical Results for Rank Bets

If soda is a Q and a bet is placed on Q from the first turn and allowed to remain until resolved, then $S = 75/2499$, $N = 0$, and $G = G_C = -75/4998 = -1.5006\ldots$ percent. (Here and in the sequel last digits are rounded off.) If instead soda is an X, then $S = 101/2499$, $N = 0$, and $G = G_C = -101/4988 = -2.0208\ldots$ percent. The player who ignores soda and then immediately places a bet on Q, letting it remain until the bet is decided, has $S = 33/833$, $N = 0$, and

$$G = G_C = -33/1666 = -1.9808\ldots \text{ percent}.$$

The value $G_C = -1.5006\ldots$ percent is the best player expectation available when placing a bet on the first turn and letting it remain until resolved. Since the opportunity arises on the first turn of every deal, *$-1.5006\ldots$ percent is a first simplistic candidate for the player's expectation.* In any case, it *is a lower bound for the player's expectation, given any reasonable playing strategy.*

2.4 Player's Expectations for Rank Bets

The question of most interest for the bet on a rank is, "What is the player's expectation?" The question is not meaningful unless we specify (1) whether we want G or G_C, and (2) whether (a) the number of turns k that the bet remains is arbitrary or (b) is always until the bet is resolved or the end, i.e., $k = t = [n/2] + 1$.

We favor G_C rather than G as the quantity of importance. The first reason is mathematical: The payoffs for a one unit bet are 1, -1, $-\frac{1}{2}$, and 0 for G. For G_C they are 1, -1, and $-\frac{1}{2}$. Thus using G_C views the bet more nearly as a Bernoulli trial. The theory of Bernoulli trials is highly developed; questions of utility theory, optional stopping, and optimal gambling strategies are perhaps more readily resolved here than in any other context. See, e.g., [16]. The second reason is practical: The probability of a bet being resolved on a single turn is generally very small. The player tends to think of the resolution of the bet as the basic entity, rather than the turn or the other procedural aspects of the game.

When $m = 2$ the maximum value of $G_C(2, n; k)$ is $G_C(2, 47; 1) = -1/190 = -0.526\ldots$ percent. This is the maximum G_C for any $m \geqq 2$, and it is unique. It is also the maximum for all negative expectation bets in the game. Thus, *$-0.526\ldots$ percent is an upper bound for the* *expectation for any strategy limited to negative expectation bets.* When $m = 3$, the maximum is

$$G_C(3, 48; 1) = -1/98 = -1.020\ldots \text{ percent}$$

and for $m = 4$ it is

$$G_C(4, 47; 1) = -3/194 = -1.564 \text{ percent}.$$

It can now be proven that: *the player who wishes to maximize his (conditional) expectation per bet that is resolved should, if he can alter his bet after each turn, limit his bets to ranks with a minimum (positive) number of cards remaining.*

2.5 Player Expectation and Strategies

An obvious way to maximize the player's expectation G_C per unit bet that is resolved is to make only zero expectation (i.e., fair) bets. To prevent this, it is a custom in some games and a rule in others, that a negative expectation bet must be made on any given deal before any fair bets.

Regard the expected loss on this first bet as a fee, f, for the chance to make fair bets. If A is the expected amount that will be bet at zero expectation and resolved, then the player's goal is to minimize f/A. Our previous results show that f is minimized by betting only on turn two, only on a rank with two cards, and by placing a minimum bet. Taking the least bet as one unit, the fee per resolved bet is $0.00526\ldots$ units. If the bet is resolved, the player from turn three on bets the maximum of M units on every fair bet.

Supposing a maximum bet of 500 units and a conservatively estimated expected number of 10 resolved zero expectation bets after turn 2, we find $A = 5{,}000$ and f/A is about 10^{-6}. This is an approximate upper bound for the house edge in Faro, given that the player must complete an unfair bet on each deal before placing fair bets.

If the foregoing strategy is impractical, others which give a small edge are practical. For instance, by betting one unit on each rank at turn one, at least one bet is resolved and fair bets may be placed from turn two. Noting that $S(m, n; 1) = (m)_2/(m+n)_2$ yields $f = 1/34 = 2.941\ldots$ percent. Thus for a one dollar minimum, the fee is a mere three cents per deal. Taking $A = 5{,}000$, the house edge is about 6×10^{-6}.

Many authors have said that proper exploitation of fair bets could reduce the house edge to an unprofitable level. However, no one seems to have quantified this level, nor realized how incredibly tiny it is.

This leads us in the next section to consider whether the analysis of nonrandom shuffling gives the player a theoretical and even a practical edge.

3. NONRANDOM SHUFFLING

3.1 A Model for Human Shuffling

Card games have been played for centuries and today hundreds are well known [8, 9]. There is an extensive literature of the heuristic, and to a lesser extent of the mathematical, analysis of these games. Virtually without exception these treatments tacitly or explicitly assume random shuffling.

Since human shuffling seems intuitively to be, and in fact is, decidedly nonrandom, it is surprising that previous work on the nonrandomness of actual human shuffling seems to consist only of [17], of [4] (which includes [3]), and [11, pp. 124–5]. It is even more surprising because the play of many games is significantly altered by the nonrandomness of human shuffling. In particular, this nonrandomness yields simple winning strategies at Blackjack, Baccarat and Faro [17]. Such knowledge also may be valuable in poker and bridge.

The general analysis of nonrandom shuffling, and its application in particular instances, is complex and extensive. We hope to present it subsequently and shall limit ourselves here to indicating how an analysis of nonrandom shuffling might proceed to develop practical favorable strategies in Faro and in Blackjack.

A common shuffle for one 52-card deck is to first cut the deck into two approximately equal parts. Then these parts are riffled together. The cut and riffle may be repeated several times. A final cut generally follows. We study this shuffle via a crude mathematical model.

Assume that $2N$ cards, labelled $1, 2, \cdots, 2N$, are cut into two equal parts. Then the cards are riffled at which time cards c_i and c_{N+i}, at locations i and $N+i$, respectively, $1 \leq i \leq N$, vie for locations $2i-1$ and $2i$. Assume that the outcome for i is with probability one half $c_i \to 2i-1$ and $c_{N+i} \to 2i$ and probability one half $c_i \to 2i$ and $c_{N+i} \to 2i-1$. Assume the outcomes are independent for distinct i. Thus the process s (*shuffle*), consisting of the cut into two equal parts followed by a riffle, may produce any one of 2^N distinct permutations, each with probability 2^{-N}.

Define the square matrix P of order $2N$ by letting the element P_{ij} be the probability that s sends c_i to location j. Then $P_{i,2i-1} = P_{i,2i} = 0.5$, $1 \leq i \leq 2N$, where all subscripts are taken modulo $2N$, and $P_{ij} = 0$ otherwise. It follows by induction that the ith row of $P^{n+1} = (q_{ij})$ is the average of the rows $2^n i, 2^n i - 1, \cdots, 2^n(i-1)+1 \pmod{2N}$ of P. Thus q_{rs} is $(\sum_i P_{is})/2^n$ where the row index i runs through the values $2^n r, 2^n r - 1, \cdots, 2^n(r-1)+1 \pmod{2N}$ in the sum, with repetitions in the summation being counted according to their multiplicity.

The maximum M over the q_{ij} will be attained by $q_{1,1}$, $q_{1,2}$, $q_{2N,2N-1}$ and $q_{2N,2N}$, which agrees with our intuitive feeling that s mixes the deck better towards the center and more poorly towards the end. We find $M = f(2^{n-1}/N)/2^n$, where $f(x) = \lceil x \rceil$ where $\lceil x \rceil$ is the "ceiling" of x [11], the integer obtained by rounding x up. If k is the greatest integer for which $2^k < N$, then P^n will contain a zero if and only if $1 \leq n \leq k$. In particular, $q_{1,2N} = 0$. The minimum m over the q_{ij} will be attained by $q_{1,2N-1}$, $q_{1,2N}$, $q_{N+1,2N-1}$, and $q_{N+1,2N}$, and has value $[2^{n-1}/N]/2^n$. When $2^n < N$, $M - m = M$. When $2^n \geq N$, $M - m = 0$

if N divides 2^n and if not, then $M - m = 2^{-n}$. This shows us the rate at which $\lim P^n$ approaches the matrix $(1/2N)$.

In particular, when $2N = 52$, P^n still contains numerous zeros after five shuffles. And for four, five and six shuffles, $M = 1/32$, $1/32$, and $3/128$, respectively. Although six shuffles eliminate the zeros from P^n, six shuffles do not produce all permutations with positive probability. However, we can find a lower bound for the number of shuffles required to do this by noting that s is a combination of 2^{26} permutations. Hence in k shuffles we can generate at most $(2^{26})^k$ permutations, and we obtain the desired lower bound by requiring that $2^{26k} \geqq (52)!$ Thus $k \geqq 9$ when $2N = 52$. (Compare [11, pp. 126–7, Exercise 13].)

This leaves the question of whether s is an asymptotically random shuffling procedure. We mean by this that if σ is any permutation and $P_\sigma(s^n)$ is the probability that n repetitions of s produce σ, then $\lim_{n\to\infty} P(s^n)$ exists and is $1/(2N)!$

For a short discussion of nonrandom shuffling, see [11, pp. 124–6] and for a brief and illuminating introduction to the notion of random sequence see [11, pp. 127–51]. Limitations on the degree of randomness as measured by "equidistributed" are given in [14] and in [11, p. 155, prob. 39]. An approach to the study of the randomness of finite sequences is given by Kolmogorov [10]. The difficulties of trying to use certain mathematical tables and functions as sources of random numbers is discussed in [18].

3.2 Favorable Player Strategies for Faro

For illustrative purposes only, suppose now that our mathematical model describes the Faro shuffling procedure. After a deck is used it consists of two equal piles whose order is known to the player. Suppose the piles are picked up and riffled according to our model. Then it is as though the piles were initially stacked one on top the other in a known initial order, and s were then applied once. Now apply s an additional number of times, for a total of n, and follow it by a final cut near the center.

Consider cards c_1 and c_2 at initial locations 1 and 2. Let $\{Y_i: 1 \leqq i\}$ be independent identically distributed Bernoulli random variables: $P(Y_i = 0) = P(Y_i = 1) = \frac{1}{2}$. Define $\{X_i: 1 \leqq i\}$ by $X_i = Y_{2i-1} + Y_{2i}$. Let $d(c_1, c_2; n)$ be the number of cards separating c_2 from c_1 after n shuffles, counting forward from c_1 ("around the end" past 52 if necessary) to c_2. Then it can be shown that

$$d(c_1, c_2; 1) = X_1, \quad d(c_1, c_2; 2) = 2X_1 + X_2,$$

and

$$d(c_1, c_2; 3) = 2^2X_1 + 2X_2 + X_3.$$

This formula fails for $n > 3$ because c_1 and c_2 interact.

Suppose now that $n \leqq 3$ and the final cut includes at least 16 cards in the top stack. Then both c_1 and c_2 will appear, in that order, in the lower part of the deck that is going to be played. Given n, which is in practice observable, we can calculate the distribution of $d(c_1, c_2; n)$.

Take $n=1$ to illustrate the ideas, and set $d = d(c_1, c_2; 1)$. Then $P(d = 0) = P(d = 2) = 1$ and $P(d = 1) = \frac{1}{2}$. When the player sees c_1 appear in play, he knows c_2 is close behind. If c_1 is the second card in a turn, then if c_2 appears on the next turn (this happens with probability $\frac{3}{4}$) then the probability is $\frac{1}{3}$ that it will be the first member of the pair and $\frac{2}{3}$ that it will be the second member. Thus the conditional player expectation on the rank Q of c_2 to win is $+\frac{1}{3}$ if it is the last Q. If c_2 fails to appear, then it will appear with certainty as the first card on the subsequent turn and the expectation of the bet on Q to lose is $+100$ percent on this turn if it is the last of its rank. The treatment for $n = 2$, 3, and for when the number of remaining Q cards is m_1 and the number of others is m_2, is evident.

It is not the details that we wish to illustrate here, but rather enough of the ideas to indicate that a practical approach is feasible.

The approach might proceed as follows. The player transmits all relevant information by radio link to a computer. He asks the computer when he wishes instructions. The computer evaluates the shuffling, and estimates the corresponding distribution functions. Observed shuffling processes s fall into a few general types, as does the final cut. For each type (perhaps for each individual) and for each n a table could be constructed giving the distribution of the distance by which c_{i+1} follows c_i for each i, $1 \leqq i \leqq 52$. Smoothing procedures and statistical decision procedures would allow one to use the data as it is accumulated, so that advantages might be available as early as the second use of the deck.

In games we have observed, used decks are sometimes placed in a wheel containing several of them. The wheel is spun and a deck is drawn at random for the next play. The uncertainty about which deck has been drawn might be overcome by constructing a separate table for each deck and using this knowledge to identify the selected deck after the first few cards appear.

We do not expect the reader to attempt to exploit the possibility of nonrandom shuffling to achieve positive expectation. Faro does not justify the effort. We simply wish to show that such possibilities are feasible in principle. The interesting mathematical aspects of nonrandom shuffling will be developed and practical applications to games of wider interest will be given in subsequent articles.

3.3 Application of Nonrandom Shuffling to Blackjack

Let $(c_1, \cdots, c_n)$ be the order of an n-card deck just prior to shuffling for the next deal. Call this the *initial order*. Thus c_i denotes the card in the ith position. For $1 \leqq i \leqq n$, define the *follower* of c_i to be c_{i+1} and define the *antecedent* of c_i to be c_{i-1}, where subscripts are modulo n.

Assume now that a shuffling procedure s, consisting perhaps of some riffles and cuts, is applied to the deck. We assume that the procedure $s = \sum_{i=1}^{n!} p_i\sigma_i$ has fixed probabilities p_i of realizing each permutation σ_i. This means that the probability characteristics of s do not change from one application of s to another, i.e., that s is time-invariant or stationary.

Let the permutation σ be a particular occurrence of s. The new order $(c_{\sigma(1)}, \cdots, c_{\sigma(n)})$ which places $c_{\sigma(i)}$ in location i, $1 \leqq i \leqq n$, is called the *final order*. This is the order in which the cards will be dealt.

Experiment: Choose an empirical procedure and record for s the following data:

Define $d(c_{\sigma(i)}, c_{\sigma(i)+1})$ to be the number of cards in the final order which separate $c_{\sigma(i)}$ from its follower in the initial order, $c_{\sigma(i)+1}$. Let

$$D(c_{\sigma(i)}, c_{\sigma(i)+1}) \equiv D_i = d(c_{\sigma(i)}, c_{\sigma(i)+1})$$

if $c_{\sigma(i)+1}$ follows $c_{\sigma(i)}$ in the final order. Let

$$D(c_{\sigma(i)}, c_{\sigma(i)+1}) \equiv D_i = -1$$

otherwise. Record D_i for each i perhaps by tallying on graph paper).

After several shuffles, a picture of the distribution of D_i given i will appear. The conditional distribution given $D_i > 0$, i.e., given that the follower of $c_{\sigma(i)}$ still follows it in the final order, is expected to be the anticipated useful predictor.

Shuffles to try:

(a) One cut near the center, followed by a riffle.
(b) Try (a) iterated k times, for $k = 2, 3, \cdots$, as indicated by results at the time.

We suspect that the plot in (a) if suitably applied (and perhaps modified by replacing D_i by the directed distance δ_i obtained by counting forward, around the end of the deck if necessary) will generate (b).

For simplicity, number the cards $1, 2, \cdots, n$, and always start with initial order $c_i = i$, $1 \leqq i \leqq n$.

A sufficiently nonrandom prediction gives a practical procedure for one deck. Note too the immediate application to Ace-location in bridge and poker.

It is becoming more common to play blackjack with two to four decks mixed together. This presents problems of indistinguishability of distinct cards. The effect is to rapidly average or smooth the distribution and "wash out" nonrandomness of our prediction.

A virtue of the method is its simplicity to apply in practice, relative to other methods. In rubber bridge, for instance, cards are played in little packets of four, called tricks. It is often convenient to note the antecedents of a key card within a trick. If the order within the trick is not disturbed then we know the antecedent of the key card in the initial order. It is generally the case that the trick order is not disturbed, and it is generally the case that one can confirm this by observation in the specific instance.

It is questionable whether we can locate the trick in the initial order well enough to use knowledge of the variation with i of the distribution of D_i. If we cannot use this knowledge, and must instead use the average distribution $\bar{D}$ of D_i, then it will depend on whether $\bar{D}$ is sufficiently nonuniform.

APPENDIX: THE LITERATURE ON MODERN FARO

Previous treatments of modern Faro appear to be erroneous or inadequate. To illustrate, Wilson [20] quotes the 1957 Encyclopedia Britannica assertion that "it is empirically estimated that the house retains 2.5 percent of the amount bet." So-called empirical estimates are not refutable, but we have seen that 1.5 percent and less is the actual state of affairs. He also quotes the 1962 Collier's Encyclopedia absurdity that "the bank's advantage is apparently at least four percent, but expert mathematicians (sic) believe it to be nearer 15 percent."

Scarne [15] claims the house edge on "splits" (defined below) is approximately two percent. In reality, the edge varies with the split, with over one hundred distinct values which range from a high of 50 percent to a low of 0.526 . . . percent if we omit unresolved bets, and from 50 percent to 0.043 . . . percent if we include them. Riddle [13] oscillates between two and zero percent for the house edge, and seems to believe that it is two percent on splits and, correctly, zero percent on ranks only one of which remain.

Collver [1] claims that for a bet at the first turn on the same rank as soda (the first card), the house edge is 0.79 percent, and for a bet on another rank, it is 1.57 percent. The bet thus described is not well defined. Is the bet left on for only one turn or until the end? Is this the edge given that the bet is resolved, or the edge without this condition? Whichever interpretation we choose, the figures cited are wrong. There are similar objections to Collver's other claims about the house edge.

Epstein [4] gives the house edge as 2.94 percent for the wager on Q to win when four Q cards remain. Actually, the figure varies depending on how many $X \neq Q$ cards also remain when the bet is placed. The range is from 1.546 . . . percent to 30 percent if we omit unresolved bets and from 0.235 percent to 30 percent if we include them. The figures of two percent when three Q cards remain, and 1.02 percent when two Q cards remain, are similarly defective. Epstein agrees and has been kind enough to check the correctness of a portion of our calculations.

Wilson computes the player expectation for a bet on a specified rank as $-1/66$, or about -1.5 percent, assuming no knowledge of soda. His usage "Suppose you are betting on Q to win. . . . As the deck is played through, eventually a Q will appear." [20, p. 216] and ". . . as you placed your bet at the beginning of the game . . ." [20, p. 217] indicates that the player places a bet at turn one and lets it remain until it is resolved. The correct value in this case is -1.9808 . . . percent.

Wilson said [21] that what he actually calculated was the player's expectation assuming he continuously maintains a constant bet on a rank until that rank is exhausted. Thus no knowledge of the already played cards is used. Observe that this procedure leads to the resolution of two, three, or four bets. The argument in [20] applies if we then interpret "player's expectation" as "player's expectation per (unit) bet resolved." The problem thus solved is of little practical interest because it assumes the player ignores (some or all of) the knowledge of which cards have already been played. But this information is always publicly and continuously displayed for the use of the players. Wilson's case of no information has this interpretation: If a turn is selected at random and a rank (to win) is selected at random, the player's expectation per resolved bet is $-1/66$ or about -1.5 percent.

As a matter of general interest, the only mathematically competent collections in English on games of chance are the excellent presentations of Epstein [4] and Wilson [20].

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