INTERNAL POINTS OF CONVEX SETS

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An interior point of a set K in a topological linear space is always an internal point of the set [1; V, 2.1(b)], as follows immediately from the definitions. Klee gives an example [3; p. 450] which shows that the converse may fail for certain subspaces of the l_p spaces, even though the set K is convex. It is the purpose of this note to describe as best we can the class of topological linear spaces for which the converse fails. We show that it fails for a large class of infinite dimensional topological linear spaces. The class includes all normed linear spaces and even all pre-F-spaces. Notation and terminology follow [1] unless otherwise indicated. If A is a set, card A designates the cardinality of A.

Theorem 1. *Suppose that a topological linear space E has a neighbourhood basis $\mathscr{U}$ at the origin O such that card $\mathscr{U}$ is less than or equal to the dimension of E. Then there is a symmetric convex set K in E having O as an internal point and having no interior points.*

Proof. We suppose first that the scalars are real. Let

$$\{h_a\}_{a \in A} \cup \{h_U\}_{U \in \mathscr{U}}$$

be a Hamel basis H for E such that h_U is in U for each neighbourhood U in $\mathscr{U}$. The existence of a linearly independent family $\{h_U\}$ with h_U in U for each U in $\mathscr{U}$ follows from the absorbing property of each neighbourhood and the condition on card $\mathscr{U}$. Let

$$K = co(\{h_a\} \cup \{-h_a\} \cup \{h_U\} \cup \{-h_U\}).$$

Then K is convex and symmetric. To see that the origin is not an interior point, we note first that no vector ch, where h is a member of the Hamel basis and $c > 1$, is in K. Suppose instead that ch is in K. It is easy to show that this is true if ch has the form $ch = \sum_{i=1}^{n} a_i h_i$, where $|a_i| \leqslant 1$, $\Sigma |a_i| = 1$, and $h_1, \ldots, h_n$ is some finite subset of H. Since h and the h_i are members of the Hamel basis, this can happen only if $h = h_j$, some j; $a_i = 0$, $i \neq j$, and $c = a_j$. But this is impossible because $c > 1$ and $|a_j| \leqslant 1$.

Since h_U is in U, ch_U is in U for sufficiently small $c > 1$. But ch_U is not in K so O is not an interior point of K. There are no interior points of K for if p were such a point, $-p$ would also be interior and therefore the line segment joining $-p$ and p would consist of interior points. Thus O would be an interior point, a contradiction.

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To see that the origin is an internal point, let x be any non-zero vector in E. Then $x=\sum_{i=1}^{m} a_i h_i$ for some finite subset $h_1, \ldots, h_m$ of H. Also, $1/\epsilon=\sum_{i=1}^{m} |a_i|>0$ since $x \neq 0$. Thus ϵx is in K. Now $-\epsilon x$ is also in K so the line segment joining them is too. This shows O is an internal point of K.

If the scalars are complex, we consider E as a real vector space of "twice" the dimension, *i.e.* choose a Hamel basis for E over the complex numbers, then work with $H \cup iH$, which is a Hamel basis for E over the reals. The proof is then the same as for the case of real scalars.

COROLLARY 2. *Every infinite dimensional normed linear space contains a convex set K and a point p in K such that p is an internal point but not an interior point.*

COROLLARY 3. *A normed linear space (more generally, any subspace of an F-space,* i.e. *a pre-F-space) is finite dimensional if the concepts of internal point and interior point coincide for convex sets.*

Proof. By Corollary 2 and the fact that an interior point is always an internal point, it suffices to prove that an internal point of a convex set K in a finite dimensional topological linear space is an internal point of K. Since all Hausdorff topological linear spaces of dimension n are equivalent, it suffices to establish the assertion for the Euclidean topology.

Suppose first that the scalars are real. Let K be a convex subset of E^n [**1**; IV, 2.1] and let p be an internal point of K. Then if $e_1, \ldots, e_n$ is a basis for E^n, there are positive numbers $\epsilon_1, \ldots, \epsilon_n$ such that

$$\{p+\epsilon_1 e_1, \ldots, p+\epsilon_n e_n\} \subset K.$$

Then $\mathscr{L}(K)$ (see [**2**; pp. 14ff] for definition and properties) equals E^n so K has an interior point [**2**; page 16, Theorem 4]. Therefore the interior and internal points of K coincide [**1**; V, 2.1(c)].

The proof extends to the complex case via the device used at the end of the proof of Theorem 1.

References

1. N. Dunford, and J. Schwartz, *Linear operators, Part I.* (Interscience, New York 1958).
2. H. G. Eggleston, *Convexity*, Cambridge Tracts in Mathematics amd Mathematical Physics, No. 47. (Cambridge University Press, London, 1958).
3. V. L. Klee, Jr. " Convex sets in linear spaces ", *Duke Math. J.*, 18 (1951), 443–466.

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