INTERNAL POINTS OF CONVEX SETS

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An interior point of a set $K$ in a topological linear space is always an internal point of the set $[1; V, 2, 1(b)]$, as follows immediately from the definitions. Klee gives an example $[3; p. 450]$ which shows that the converse may fail for certain subspaces of the $l_n$ spaces, even though the set $K$ is convex. It is the purpose of this note to describe as best we can the class of topological linear spaces for which the converse fails. We show that it fails for a large class of infinite dimensional topological linear spaces. The class includes all normed linear spaces and even all pre-$F$-spaces. Notation and terminology follow [1] unless otherwise indicated. If $A$ is a set, card $A$ designates the cardinality of $A$.

Theorem 1. Suppose that a topological linear space $E$ has a neighbourhood basis $\mathcal{U}$ at the origin $O$ such that card $\mathcal{U}$ is less than or equal to the dimension of $E$. Then there is a symmetric convex set $K$ in $E$ having $O$ as an internal point and having no interior points.

Proof. We suppose first that the scalars are real. Let

$$\{h_u\} \cup \{h_v\}$$

be a Hamel basis $\mathcal{H}$ for $E$ such that $h_u$ is in $U$ for each neighborhood $U$ in $\mathcal{U}$. The existence of a linearly independent family $\{h_u\}$ with $h_u$ in $U$ for each $U$ in $\mathcal{U}$ follows from the absorbing property of each neighborhood and the condition on card $\mathcal{U}$. Let

$$K = \omega(\{h_u\} \cup \{h_v\} \cup \{-h_u\})$$

Then $K$ is convex and symmetric. To see that the origin is not an interior point, we note first that no vector $ch$, where $c$ is a member of the Hamel basis and $c > 1$, is in $K$. Suppose instead that $ch$ is in $K$. It is easy to show that this is true if $ch$ has the form $ch = \sum a_i h_i$, where $a_i \leq 1$, $\Sigma |a_i| = 1$, and $h_1, ..., h_n$ is some finite subset of $\mathcal{H}$. Since $h$ and the $h_i$ are members of the Hamel basis, this can happen only if $h = h_i$, some $j$; $a_i = 0$, $i \neq j$, and $c = a_j$. But this is impossible because $c > 1$ and $|a_j| \leq 1$.

Since $h_u$ is in $U$, $ch_u$ is in $U$ for sufficiently small $c > 1$. But $ch_u$ is not in $K$ so $O$ is not an interior point of $K$. There are no interior points of $K$ for if $p$ were such a point, $-p$ would also be interior and therefore the line segment joining $-p$ and $p$ would consist of interior points. Thus $O$ would be an interior point, a contradiction.

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To see that the origin is an internal point, let \( x \) be any non-zero vector in \( E \). Then \( x = \sum_{i=1}^{n} a_i h_i \) for some finite subset \( h_1, ..., h_m \) of \( H \). Also, 
\[ \frac{1}{\epsilon} = \frac{1}{\epsilon} \sum_{i=1}^{n} |a_i| > 0 \] since \( x \neq 0 \). Thus \( \epsilon x \) is in \( K \). Now \(-\epsilon x\) is also in \( K \) so the line segment joining them is too. This shows \( O \) is an internal point of \( K \).

If the scalars are complex, we consider \( E \) as a real vector space of "twice" the dimension, i.e. choose a Hamel basis for \( E \) over the complex numbers, then work with \( H \oplus iH \), which is a Hamel basis for \( E \) over the reals. The proof is then the same as for the case of real scalars.

**Corollary 2.** Every infinite dimensional normed linear space contains a convex set \( K \) and a point \( p \) in \( K \) such that \( p \) is an internal point but not an interior point.

**Corollary 3.** A normed linear space (more generally, any subspace of an F-space, i.e. a pre-F-space) is finite dimensional if the concepts of internal point and interior point coincide for convex sets.

**Proof.** By Corollary 2 and the fact that an interior point is always an internal point, it suffices to prove that an internal point of a convex set \( K \) in a finite dimensional topological linear space is an internal point of \( K \). Since all Hausdorff topological linear spaces of dimension \( n \) are equivalent, it suffices to establish the assertion for the Euclidean topology.

Suppose first that the scalars are real. Let \( K \) be a convex subset of \( E^n \) \([1; 14, 2.1]\) and let \( p \) be an internal point of \( K \). Then if \( e_1, ..., e_n \) is a basis for \( E^n \), there are positive numbers \( \epsilon_1, ..., \epsilon_n \) such that
\[ \{p + \epsilon_1 e_1, ..., p + \epsilon_n e_n\} \subseteq K. \]

Then \( \mathcal{L}(K) \) (see \([2; pp. 14ff]\) for definition and properties) equals \( E^n \) so \( K \) has an interior point \([2; page 16, Theorem 4]\). Therefore the interior and internal points of \( K \) coincide \([1; V, 2.1(e)]\).

The proof extends to the complex case via the device used at the end of the proof of Theorem 1.

**References**


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