FINITE DIMENSIONAL NORMED SPACES†

E. O. Thorp

Several characterisations of finite dimensional normed spaces are well known. The usual characterisation is some variant of the theorem of F. Riesz [1; Theorem IV 3.5].

A normed linear space is finite dimensional if and only if its closed unit sphere is compact. Numerous additional conditions may be given. For example, it is well known (see, *e.g.* [4; p. 190, Ex. 1, and p. 193, Ex. 1]) that *a normed linear space is finite dimensional if and only if its conjugate is finite dimensional.* Another condition is that *a normed linear space is finite dimensional if and only if every linear functional is continuous.* The proof is straightforward. We give below two characterisations which we believe are new.‡

Theorem 1. *A normed linear space X is finite dimensional if and only if its image TX is closed whenever T is a continuous linear mapping of X into a normed space.*

Proof. It is well known that finite dimensional spaces remain finite dimensional under linear maps and that finite dimensional subspaces of normed linear spaces are closed. Therefore it suffices to prove that if a normed space has the stated property, then it is finite dimensional.

If, instead, X is infinite dimensional and has the property, then X must be complete. Otherwise, it can be isometrically imbedded as a dense proper subspace of its completion, contradicting the hypothesis. The set $B(X)$ of bounded linear operators from X into itself is therefore complete in the operator norm topology. The subspace $B_c(X)$ of compact linear operators is known to be a closed (therefore complete) subspace.

Using well-known techniques we show that there is a compact operator in $B_c(X)$ with infinite dimensional range. Define a sequence $\{P_n\}$ of one-dimensional operators of norm 1 as follows. Choose x_1 in X such that $\|x_1\| = 1$. Let $f_1(cx_1) = c$ for all scalars c. By the Hahn-Banach theorem there is a continuous extension F_1 of f_1 to all of X. Define P_1 by

$$P_1(x) = F_1(x)\,x_1/\|F_1\|$$

for all x in X. The norm of P_1 is evidently 1. Continuing by induction,

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choose x_n in X such that $\|x_n\|=1$ and $P_j(x_n)=0$ for $j=1, \ldots, n-1$. This is possible because X is infinite dimensional and the codimension of $\{x: P_j(x)=0, j=1, \ldots, n-1\}$ is less than or equal to $n-1$. Let $f_n(c_1x_1+\ldots+c_nx_n)=c_n$. Let F_n be a continuous extension of f_n to all of X. Define P_n by $P_n(x)=F_n(x)x_n/\|F_n\|$ for all x in X. The norm of P_n is evidently 1.

The series $\sum_{n=1}^{\infty} P_n/2^n$ converges in the operator norm to a compact operator P since $B_c(X)$ is complete. The range of P contains $\{x_n\}$, since

$$x_n=\left(\sum_{i=1}^{n+m} P_i/2^i\right)(2^n\|F_j\|x^n)\to P(2^n\|F_j\|x^n)$$

as $m\to\infty$. Therefore the range of P is infinite dimensional. By our hypothesis, the range of P is closed. This contradicts the well-known fact that the range of a compact operator cannot contain an infinite dimensional closed subspace. To see this, suppose T is compact and the range $R(T)$ of T contains the closed subspace F. Then if T_0 is the restriction of T to $E=T^{-1}(F)$, T_0 is a compact map from E onto the Banach space F. If N is the null manifold of T_0, then the induced operator T_1 from E/N onto M is compact, one-to-one, and onto. It is an isomorphism by the closed graph theorem, hence F is finite dimensional.

DEFINITION. *A normed linear space X has property (D) if and only if for every collection $\{M_a\}_{a\in A}$ of nested dense linear manifolds, $\bigcap_{a\in A} M_a$ is dense in X.*

THEOREM 2. *A normed linear space is finite dimensional if and only if it has property D.*

Proof. If X is finite dimensional, X is linearly homeomorphic to E^n, whence it is clear that the only dense manifold is X itself, therefore X has property (D).

If X is not finite dimensional, we show X does not have property (D). Suppose it did. Then an easy application of Zorn's lemma shows that there is a minimal dense linear manifold M in X. Choose any $h\neq 0$ in M and extend $\{h\}$ to a countable bounded linearly independent subset $h, h_1, h_2, \ldots, h_n, \ldots$ of M. Consider the sequence defined by $h'=h$, $h_1'=(1+1)h+h_1, h_2'=\left(1+(\tfrac{1}{2})\right)h+(\tfrac{1}{2})h_2, \ldots, h_n'=\left(1+(1/n)\right)h+(1/n)h_n, \ldots$ The sequence $h', h_1', \ldots, h_n', \ldots$ is linearly independent and therefore can be extended to a Hamel basis $H'=\{h_b'\}_{b\in B}$ for M. Now

$$\lim_n (h_n'-h)=\lim_n (1/n)(h_n+h)=0$$

so M', the manifold spanned by $H'-h$†, is dense in M and therefore in X. Yet $M' \underset{\neq}{\subset} M$, contradicting the hypothesis that M is minimal.

Remark. The proof shows that there are no minimal dense subspaces of an infinite dimensional normed space. It also shows that in the step from a manifold M to a manifold M', dense in M and with codimension 1 relative to M, any one element of M can be excluded from M'.

The question naturally arises as to what extent Theorems 1 and 2 hold in topological linear spaces. A partial answer for Theorem 2 is given below.

THEOREM 3. *If E is a topological linear space which has a dense Hamel basis, E does not have property (D). Hence Theorem 2 extends to such spaces.*

Proof. Let $H = \{h_b\}_{b \in B}$ be a dense Hamel basis. Let $\mathscr{M}$ be the collection of manifolds which are both dense and are the span of some subset of H. Partially order $\mathscr{M}$ by inclusion. Suppose each totally ordered subset $\{M_a\}_{a \in A}$ has an upper bound B in $\mathscr{M}$. Then B is dense, hence $M = \bigcap_{a \in A} M_a$ is dense because $M \supset B$. Also M is the span of the elements of H which are common to all the M_a so $M \in \mathscr{M}$. Therefore M is an upper bound for $\{M_a\}$.

From Zorn's lemma it follows that there is a minimal element N of the set $\mathscr{M}$. The elements of H which span N are therefore dense. If we delete one of them, the remainder are still dense, and span a smaller dense manifold N', contradicting the minimality of N. Thus some totally ordered subset $\{M_a\}_{a \in A}$ does not have an upper bound B in M. But since M is spanned by a subset of H and is contained in each M_a, and M fails to be an upper bound, M is not dense. Therefore E does not have property (D).

Remarks. Klee has shown that if E has a neighbourhood basis at the origin with cardinality less than or equal to the dimension of E, then E admits a dense Hamel basis. A proof of this theorem appears in [**2**; p. 448].

Theorem 2 is consequently a corollary of Theorem 3. However, the simpler shorter proof given may be of some interest in itself.

Not all topological linear spaces admit a dense Hamel basis nor do all topological linear spaces satisfy Theorem 2. Let X be any infinite dimensional vector space and let X^+ be the set of all linear functionals on X. Then the X^+ topology for X [**1**; p. 419] makes X a locally convex topological linear space. It can be shown that *all* subspaces of X are closed in the X^+ topology. Thus the only dense manifold is X itself so the space has property (D). No Hamel basis is dense for if one were, we could omit one element and the hyperplane spanned by the remaining elements would be dense and proper hence not closed.

† here $H'-h$ denotes the set of elements H' less the element h.

References

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New Mexico State University,
University Park, New Mexico, U.S.A.