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SEPARATUM

Source of variation	Sum of squares	d.f.	Mean squares	F-ratios
Quadratic regression	112 967.13	1		517***
Additional for linear term	1 104.75	1		5.06*
Additional for constant term	6.40	1		0.03
Lack of fit	4 059.92	16	253.75	
Within groups	6 764.80	31	218.22	

$$A_1X_1/D_1 = 0.1533\ x^2$$
$$A_2X_2/D_2D_1 = -0.0632\ x^2 + 1.2390\ x$$
$$\hat{y} = 0.0901\ x^2 + 1.2390\ x$$

SUMMARY

Multiple linear regression functions can be fitted by means of orthogonal polynomials. This method allows to include successively additional predictor variables in a prespecified order without the necessity of recomputing the regression coefficients of all previously included predictors. The orthogonal terms and the sums of squares for the analysis of variance can easily be computed from determinants constructed from the sums of squares and products of the observations.

RÉSUMÉ

L'adaptation des fonctions de régression multiple et linéaire peut être réalisée à l'aide de polynomes orthogonaux. La méthode permet d'inclure les variables explicatives successivement et de faire des tests de signification pour chaque variable supplémentaire. Les fonctions orthogonales de régression et les sommes des carrés de l'analyse de variance peuvent être calculées à l'aide de déterminants composés des sommes des carrés et produits des observations.

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EXTENSIONS OF THE BLACK-SCHOLES OPTION MODEL*

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Summary. The Black-Scholes option theory is a breakthrough. It has simple hypotheses, provides a valuation formula using only observables, and explains actual prices. We solve the corresponding problem for the two types of warrant hedges. Black and Scholes assume no cash dividends. For one cash dividend we give the condition for the solution to be unchanged. When the dividend affects the solution, we give upper and lower bounds. The method extends to a finite series of cash dividends.

1. Introduction. An option on common stock (ordinary shares) is the right to buy (a call) or to sell (a put) a specified number of shares (usually 100) at a specified price (striking price) until a specified time (expiration date). A European option may be exercised only at expiration. An American option may be exercised at any time before expiration.

The common stock purchase warrant is an option similar to the call option. Warrants are issued by the company who possesses the stock the warrants claim. Terms typically are of the form A warrants + B dollars obtain C shares of stock until time t^*.

Let $x(t)$ be the stock price at time t, c the exercise price of the option, t^* the expiration time of the option, $w(t)$ the price of a call on one share and $p(t)$ the price of a put on one share. If $y+ = \max(y, 0)$ and $y^- = \max(-y, 0)$, it is evident that for American options and $t \leqq t^*$, $w(t) \geqq (x(t)-c)^+$, $p(t) \geqq (x(t)-c)^-$, and that $w(t)-p(t) \geqq x(t)-c$. Many other functional relationships are also evident or strongly suggested. The functional dependence of option prices on the price of the underlying common has led to formulas

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for the option price as a function $f(x(t), t^*-t, \cdots)$ of stock price, time t^*-t until expiration, and a few other variables. In particular, the hope (successful!) has been to eliminate most or all of the variables which affect common stock price, by lumping them all in $x(t)$ itself.

The extensive modern mathematical theory of options was initiated by Bachelier (1900). This landmark paper developed the theory of Brownian motion and applied it to option prices. It assumed stock price changes were normally distributed (modern statistical work shows that the lognormal distribution gives a better fit) and identified volatility v (i.e., standard deviation per unit time) as a principal determinant of option price, entering the functional expression in the form $v^2(t^*-t)$.

Subsequent mathematical and statistical work developed along several lines. Regression techniques were used to construct models of past price behavior (Kassouf, 1965; Shelton, 1967, and others). Rational (normative) theories of option prices were developed by, e.g., Samuelson (1965), Samuelson and Merton (1969), and others. The lognormal model for stock prices yielded a natural expression for the price of an option in terms of the normal distribution. See, e.g., Sprenkle (1961), Harbaugh (1965), Thorp (1969), and others. Hedging techniques (e.g., call options or warrants short, stock long) as in Thorp and Kassouf (1967), and Thorp (1971) give information about the use of options in optimizing portfolio performance.

2. The Black-Scholes theory. These efforts have culminated in a breakthrough by Black and Scholes (1972, 1973). We sketch their theory for calls. It is similar for puts and for straddles.

Black and Scholes observe that hedging options short and common stock long and continuously adjusting the mix leads to a riskless rate of return r. They argue that in market equilibrium this must equal the riskless rate of return in the market.

They further assume:

(a) The short term interest rate r is known and constant.

(b) The distribution of possible stock prices at the end of any finite interval is lognormal. The stock price follows a random walk in continuous time with variance proportional to the square of the square of the stock price. The variance rate of the return on the stock is constant.

(c) The stock pays no dividends or other distributions.

(d) The option is "European".

(e) There are no transaction costs.

(f) It is possible to borrow at rate r any fraction of the price of a security to buy it or to hold it.

(g) A short seller receives the price of the security from a buyer, and will settle on some future date by paying the price of the security on that date.

From this they show that the call price w satisfies the partial differential equation

$$(1) \qquad \frac{1}{2} v^2 x^2 w_{11} + rxw_1 + w_2 - rw = 0$$

subject to the boundary conditions

$$(2) \qquad \begin{aligned} w(x, t^*) &= x - c \quad \text{if } x \geqq c \\ w(x, t^*) &= 0 \quad \text{if } 0 \leqq x < c \end{aligned}$$

where $x \geqq 0$, $t \leqq t^*$, v^2 is the variance rate of return on the stock per unit time (say one year), and r is the riskless rate of return. Black and Scholes estimate v^2 by

$\Sigma_{i=1}^{n} \left[\left\{ x(t_i) - x(t_{i-1}) \right\} / x(t_{i-1}) \right]^2$ where $t_n - t_0 = 1$ year and the $x(t_i)$ are stock prices on successive market days. They use the 6 month commercial paper rate for r. Equation (1)-(2) is a version of the heat transfer equation of physics and is solved by Black and Scholes by standard methods to yield:

$$(3) \qquad w(x, t) = xN(d_1) - ce^{r(t-t^*)} N(d_2)$$

$$d_1 = \left[\ln(x/c) + (r+v^2/2)(t^*-t) \right] / v\sqrt{t^*-t}$$

$$d_2 = \left[\ln(x/c) + (r-v^2/2)(t^*-t) \right] / v\sqrt{t^*-t}$$

where $\ln$ is $\log_e$ and $N(d)$ is the cumulative normal density function.

Formula (3) has the virtue of containing only quantities which are given such as c and t^*, available such as t and x, or which can be estimated from data, such as r (high accuracy) and v ("fair" accuracy). The stock enters only through x and v.

Black and Scholes (1972) demonstrate via a statistical study of more than 5000 options, that their model describes actual option prices.

3. The Black-Scholes assumptions. Assumptions (a)-(g) have been examined carefully. Merton (1973) considers the extension of (a) to a stochastic interest rate. The lognormality of stock prices, with constant variance, as in (b), has been a central theme in the literature for 70 years (Cootner, 1964). It will be shown below that when the stock pays dividends (contrary to assumption (c)), the value of a call option is equal to, or can often be estimated from, the model value for no dividends. See also Merton (1973).

Until recently, most American calls had their striking price reduced for cash dividends. The recently created Chicago Board Option Exchange (C. B. O. E. now trades options which do not change their exercise price for cash dividends. Also, warrants do not. It has also been shown that for calls, but not for puts, the model price is the same for American and for European (assumption (d)) options: the optimal strategy is to hold a call (on a random-walk stock) until expiration.

Assumption (e) of no transactions costs may be nearly true in practice for certain stock exchange firms. Assumption (f) also may be nearly true in practice. Even in portfolios where borrowing is limited or is not allowed (most mutual funds, pension funds) unlimited borrowing may occur in a "virtual" way. For instance if the portfolio has some commercial paper and wishes to buy a lesser value of stock, it can finance as much as it wishes of the purchase by selling commercial paper. In terms of money, this is equivalent to buying the stock by borrowing the desired amount at the commercial paper rate.

Assumption (g), which includes tacitly the assumption that short selling is done, migh be challenged on the grounds that most market participants are either limited in the amount of short selling they may do, can do none at all, or choose to do none. The idea of "virtual" short sales answers this. Consider for instance a large portfolio which has T stock. If we sell T stock and buy T options, we have done the equivalent of adding to the portfolio a package which consists of: sell short some T stock, receiving the proceeds, and buy some T options. Thus investors who do no short selling, but instead switch from stock to option or from option to stock, can instead be regarded as having added to their original portfolio option hedges of the type used to derive the model.

4. Extension to warrant hedging. Thorp and Kassouf (1967) showed that warrant hedging (sell short appropriate expiring warrants, buy common, use strategies for determining and adjusting the proportions) led to unusually high rates of return and unusually low risk (variance).

However, assumption (g) is generally not correct for the warrant hedger. When warrants or other securities are sold short, the brokerage firm which lent the securities retains the proceeds. The short-seller loses the use of the money. Thus to sell warrants short buy stocks, and yet achieve the riskless rate of return r requires a higher warrant short sale price than for the corresponding call.

Reasoning like Black and Scholes (1973), we find that the price $\bar{w}$ which the warrant satisfies is:

$$(\bar{1}) \qquad \frac{1}{2} v^2 x^2 \bar{w}_{11} + r x \bar{w}_1 + \bar{w}_2 = 0$$

subject to the boundary conditions

$$(\bar{2}) \qquad \begin{aligned} \bar{w}(x, t^*) &= x - c \quad \text{if } x \geqq c \\ \bar{w}(x, t^*) &= 0 \quad \text{if } 0 \leqq x < c. \end{aligned}$$

The solution is

$$(3) \qquad \bar{w}(x,t) = e^{r(t^*-t)} w(x,t) = e^{r(t^*-t)} xN(d_1) - cN(d_2)$$

and $\bar{w} > w$ as expected.

The solution may be obtained in the usual way but an easier method (and the one used) is to guess the solution by (plausibly) arguing that since the short sale proceeds are not available until $t = t^*$ we expect the present value $e^{r(t-t^*)}\bar{w}(x,t)$ to equal $w(x,t)$. The solution is then verified by substitution. Remember that in obtaining this solution we have assumed that warrants and calls are the same except for (g). Black and Scholes (1973) point out differences. In the second kind of warrant hedge the warrant is purchased, the stock is sold short, and the proceeds of the short sale are again retained by the broker, rather than being used to the short seller.

For this hedge, the warrant price $\underline{w}(x,t)$ satisfies

$$(\underline{1}) \qquad \frac{1}{2} v^2 x^2 \underline{w}_{11} + \underline{w}_2 - r\underline{w} = 0$$

subject to the boundary conditions

$$(\underline{2}) \qquad \begin{aligned} \underline{w}(x, t^*) &= x - c \quad \text{if } x \geqq c \\ \underline{w}(x, t^*) &= 0 \quad \text{if } 0 \leqq x \leqq c. \end{aligned}$$

Again the plausible guess yields the solution. The short sale proceeds of the stock will not be available until time t^* so the present value of the proceeds are $e^{r(t-t^*)}x$. This suggests

$$(\underline{3}) \qquad \underline{w}(x,t) = w\left(e^{r(t-t^*)}x, t\right)$$

which is readily verified. Since w is strictly increasing in x, $\underline{w} < w$.

For a given t consider the $\underline{w}(x,t)$, $w(x,t)$ and $\bar{w}(x,t)$ curves in the (x, w) plane. The $\underline{w}$ and $\bar{w}$ curves define a strip which contains the w curve. If the market were efficient transactions would occur close to the w curve and generally within this band. Hedgers would only operate outside the band so hedging would never occur. But the option markets are inefficient and hedging does occur.

5. Calls on stocks paying cash dividends. Let t_1 be the last market time that the owner of a stock is entitled to a particular dividend D and let $t_2 > t_1$, be the first subsequent market time that he is not. Then, ceteris paribus, $x(t_1) = x(t_2) + D$. Let D be the only dividend before the expiration of a call on the stock. If we plot (x, w) for $t \leqq t_2$ on axes shifted right an amount D, the B-S option price at t_1 will be $\max(w(x,t_1), x-c)$. For $w(x,t_1)$ we use the model price (3) relative to the right-shifted axes.

If this curve $w(x,t_1)$ intersects $w = x-c$, say at (x^*, x^*-c), then the option should be exercised at t_1 exactly when $x(t_1) > x^*$. In this case, the model price $w_D(x,t)$ for the option will satisfy $w_D(x,t) > w(x,t)$ when $t \leqq t_1$. If $w(x,t_1)$ does not intersect $w = x - c$, and there is just the one dividend D to consider, then $w_D(x,t) = w(x,t)$ for all $t \leqq t^*$. The curves (3) are above and asymptotic to $w = x - ce^{r(t-t^*)}$ from which it follows that intersection occurs if and only if $r(t-t^*) > \ln(1-D/c)$ or (6) $t > t^* + r^{-1}\ln(1-D/c)$.

A sufficient condition for this is $r(t^*-t) < D/c$ (Merton, 1973).

As an example, consider the C.B.O.E. April 50 call option for T (American Telephone and Telegraph. We have t^* = close, April 30, 1973, t_1 = close, February 21, 1973, $t^*-t_1 = 68/365$, $c = \$50$ and $D = \$0.70$. With $r = 7.5\%$, $r(t-t^*) = -.01397 > -.01410$ so intersection (barely) occurs. If, as was the case later in the year, r were 8.0%, $r(t-t^*) = -.01490 < -.01410$ and intersection would not occur. This happened for the July 50 C.B.O.E. call options on T.

We have the lower bound $w(x,t)$ for $w_D(x,t)$. An upper bound is also readily obtained. Find the t^1 which gives equality in (6). If $t_1 \leqq t^1$, $w_D(x,t) = w(x,t)$. If $t_1 > t^1$, then intersection does not occur for an option which expires at time $t^* + t_1 - t^1$. Thus for such an option, $\tilde{w}_D = \tilde{w}$. Also $\tilde{w}_D \geqq w_D$ and $\tilde{w}(x,t) = w(x, t+t_1-t^1)$ so $w(x, t+t_1-t^1) > w_D(x,t) > w(x,t)$. In the case of the April 50 T call options, $t_1 - t^1 = 0.6$ days and the approximation is very good. In any case, $\lim_{t\to\infty}\left[w(x, t+t_1-t^1) - w(x,t)\right] = 0$ uniformly in x so $w_D(x,t) \to w(x,t)$ uniformly in x as t increases.

By applying these arguments recursively, upper and lower bounds can be obtained for $w_{D_1, D_2, \ldots}$, the model price for the option in the case of a series of cash dividends whose magnitudes are known with certainty. In particular if intersection does not occur for the last dividend in a series of equally spaced cash dividends of equal amounts, it does not occur for earlier ones. Thus in this case $w_{D_1, D_2, \ldots} = w$ for all $t \leqq t^*$. It is also true in this case that (approximately) intersection will not occur for dividends prior to the last unless the dividend rate exceeds r. This is seldom the case.

Resumé. La théorie des options de Black et Scholes a les hypothèses elegantes. Elle donne une formule pour le prix qu'il faut seulement les observables, et qui explique les prix de la bourse. Nous résoudons le problème analogue pour les "warrants" parient pour et contre. Black et Scholes supposent que il n'y a pas des dividendes d'argent. Pour un dividend d'argent nous donnons une solution partielle. La mèthode s'étend à plusieurs dividends d'argent.

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ON EXCHANGEABLE PRIOR INFORMATION IN SAMPLING FINITE POPULATIONS

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1. Summary

In the paper we introduce a model for Bayesian analysis of a finite population. The model is based on the assumption that the values of the population elements may be considered as exchangeable random variables with density $f_N(y_1,y_2,\cdots,y_N) = \int \Pi f(y_i|\omega)dW(\omega)$.

It is shown that the posterior density of the population characteristics after sampling is of the same form of the prior. For densities $f(\cdot|\omega)$ admitting one-dimensional sufficient statistics the posterior density depends only on the sufficient statistic for the sample.

Finally we give the distribution of the sample and the posterior distribution of the population for some well-known prior distributions.

2. The Model

We consider a population consisting of N elements with unknown characteristics $Y_1,Y_2,\cdots,Y_N$ and assume that a sample of n elements drawn without replacement has yielded the values $x_1,\cdots,x_n$. Thus the sample gives complete information about the particular population elements which are included in the sample, and leaves the statistician with the problem of making inference about the non-sampled individuals.