The Mathematics of

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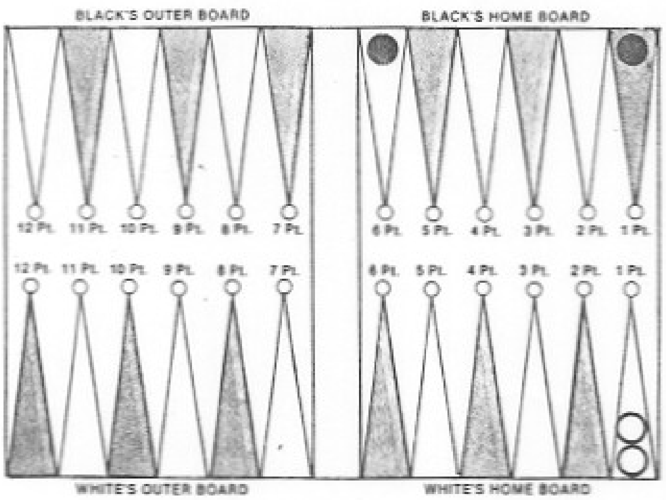
Editor's Note: Due to problems in printing the magazine, last month's "Mathematics of Gambling" may have proved confusing to some readers. In the diagram on page 44 of the October issue, readers should note that there are four *Black men on the #1 point. These men are undistinguishable due to the quality of the printing. In the diagram on page 45, there are* no *Black men on the #1 point, and there is* one *Black man on the #3 point and one on the #4 point.*

Since this month's column refers back to the tables that accompanied last month's column, we are reprinting Table 5 at the end of this article. Please note that the shading on this chart differs from that in last month's column. Table 5 appears as it appears in this month's issue is correct; *if you will go back over Dr. Thorp's column for October, you will find that it reads correctly with regard to the revised table.*

This month we will illustrate and explain the use of the tables presented last month. In that article, I referred to a book entitled *How Good Are You at Backgammon: 75 Challenging Test Situations* by Nicolaos and Vassilios Tzannes, Simon and Shuster, 1974. Consider first Situation 74 from the Tzannes' book. This is shown in Diagram 1. -

DIAGRAM 1



It is Black's turn so he is Player One. Black doubles. Should he? If he does, should White accept? The cube is in the middle. We look in Table 5, row 6+1, column 1+1. Black should not accept. If he does, White should accept. (This is correctly recommended by the Tzannes' book.) Table 3 shows that Black's expectation under best play, which means not doubling, is −17%. If instead Black has the cube, we use Tables 4 and 5. In this example we get exactly the same answer. This isn't always the case, though, as we will see.

This example is also easy to analyze directly. If Black bears off in his next turn he will win. The chances are 15/36 (Table 1, column before last). If he does not bear off at once, White will win and Black will lose. So if the current stake is 1 unit, and Black does not double, Black's expected gain is +1 unit × 15/36−1 unit × 21/36 = −6/36 = −16 2/3%. Now suppose Black doubles and White accepts. Then Black's expected gain is +2 units × 15/36−2 units × 21/36 = −12/36 = −33%. On average Black will lose an extra 16 2/3% of a unit if he makes the mistake of doubling and White accepts.

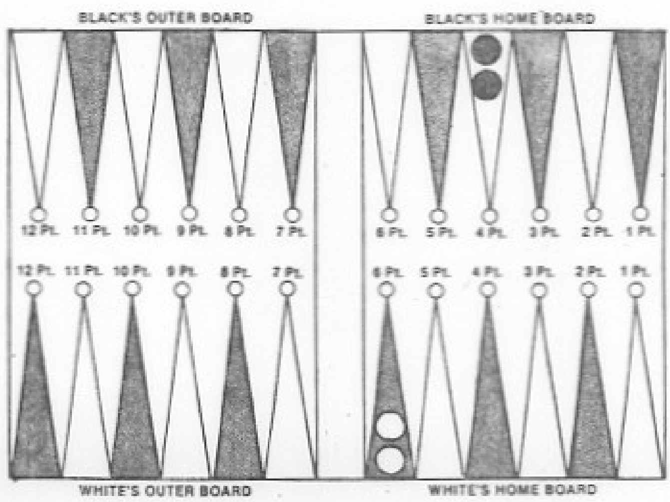
It's easy to see from this type of reasoning that if Player One has any two-man position and Player Two will bear off on the next turn, then Player One should not double (if he can) when his chance to bear off in one roll is less than 50%. If his chance to bear off is more than 50%, he should double. Referring to the same Table 1 of the article before last proves this rule which the Tzannes cite for these special situations:

With double three, six-one, six-two (or anything worse)
Keep dumb, hope for the best. Anything better, don't delay,
Double the stakes with zest.

The Tzannes' Situation 73 is similiar.

Here is a trickier situation that I don't think you could figure out without help from last month's tables. Suppose White has 6+6, Black has 4+4, White is to roll and the doubling cube is in the middle. This is shown in Diagram 2. Should White double? How does the game proceed for various rolls?

DIAGRAM 2



Gambling

End Positions in Backgammon, Part III

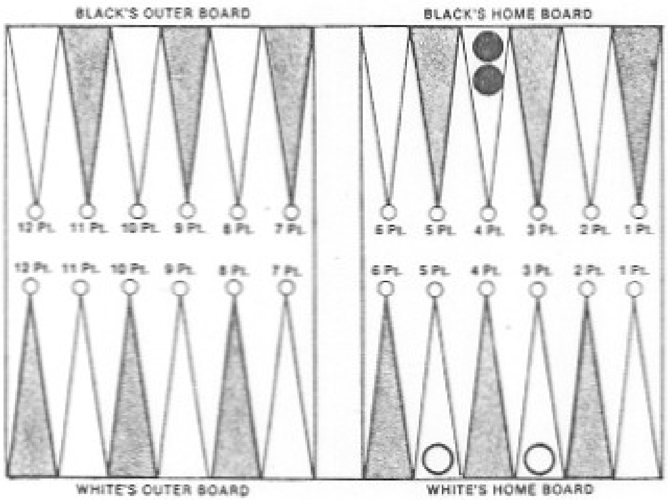
by Edward Thorp

White is Player One. He consults Table 3 from last month and sees his expectation is 16%. But Table 5 tells White not to double. We now show how to use the table to play optimally for a sample series of rolls. Suppose White rolls 3-1. How does he play it? He can end up with 6+2 or with 5+3.

The rule from two months ago says that 5+3 looks better because it gives him a greater chance to bear off on the next turn. This is proven by the tables as follows: after White plays, it will be Black's turn. Black will be Player One with 4+4, White will be Player Two with either 5+3 or 6+2. The cube will be in the middle. Which is best for White? Consult Table 3. We find Player One (Black) has an expectation of 88% if White has 5+3 whereas Black has 96% if White has 6+2. White wants to keep Black's expectation down so he plays to leave 5+3.

The situation after White makes this move is shown in Diagram 3.

DIAGRAM 3

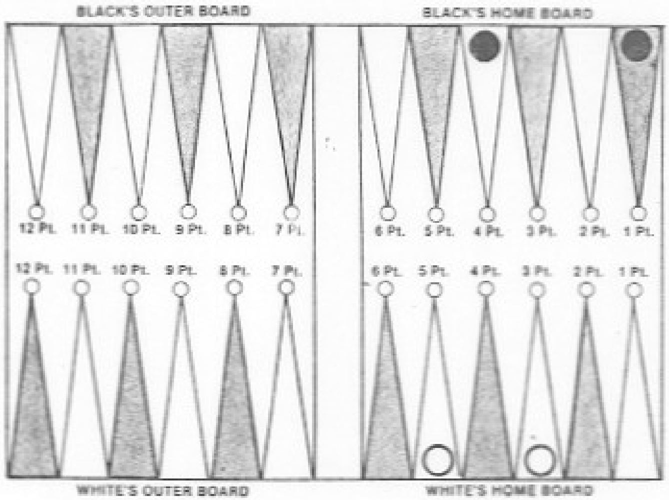


Black is to roll and the cube is in the middle. Should Black double? Should White accept? Table 5 says Black should double and White should accept. Table 3 says Black's expected gain is 88% of the one-unit stake.

Next Black rolls 2-1. He can leave 4+1 or 3+2. The rule from two months ago says 4+1 is better. To confirm this, note that after Black moves, White will be Player One with 5+3, Black will be Player Two with either 4+1 or 3+2 and White will have the cube. Therefore we consult Table 4, not Table 3. If Black leaves 4+1, White's expectation is 2% of the current two-unit stake. If Black leaves 3+2, White's expectation is 15%. Therefore Black leaves 4+1.

It is now White's turn. The situation is shown in Diagram 4.

DIAGRAM 4



The stake is 2 units, White's expectation is 2% of 2 units or .04 unit and White has the cube. What should he do? Table 5 tells us White should not double.

White now rolls 5-2, leaving 1+0. Black does not have the cube. Table 2 gives his expectation as 61% of 2 units or 1.22 units. He wins or loses on this next roll.

The tables show certain patterns that help you to understand them better. For instance, for a given position it is best for Player One to have the cube. It is next best for Player One if the cube is in the middle and it is worst for Player One for Player Two to have the cube. Therefore for a given position, Player One's expectation is greatest in Table 4, least in Table 2, and in between in Table 3. For instance, with Player One having 6+6 and Player Two having 4+4, Player One's expectation is 25% if he has the cube, 16% if it is in the middle, and 7% if Player Two has the cube.

Sometimes two or even all of the expectations are the same. For instance, if Player One has 6+6 and Player Two has 6+5, Player One's expectation is 71% if he has the cube or if it's in the middle. If Player Two has the cube Player One's expectation drops to 36%.

Examination of the doubling strategies in Table 5 shows that the positions where Player One should double and Player Two should fold are the same whether Player One has the cube or the cube is in the middle. Although this happens for the two-man end positions we are analyzing here, it is not always true in backgammon. The positions where Player One should double and it doesn't matter if Player Two accepts or folds also are the same in Table 5. But some of the positions where Player One should double and Player Two should accept are different. If Player one

continued on page 75

continued from page 29

has the cube, Table 5 shows that he should be more conservative. Intuitively, this is because if he has the cube and does not double, he prevents Player Two from doubling, whereas if the cube is in the middle, Player Two cannot be prevented from doubling.

Table 5 leads to an example that will confound the intuition of almost all players. Suppose Player One has 5+2 and has the cube. Consider two cases: (a) Player Two has 1+0 and (b) Player Two has 6+0. In which of these cases should Player One double? Clearly 6+0 is a worse position than 1+0. And the worse the position the more likely we are to double, right? So of the four possible answers (double 1+0 and 6+0, double 1+0 but not 6+0, double 6+0 but not 1+0, don't double 1+0 or 6+0) we "know" we can eliminate "double 1+0, don't double 6+0," right? WRONG. The only correct answer, from Table 5, is: double 1+0 but don't double 6+0. Try this on your expert friends. They will almost always be wrong. If they do get it right they probably were either "lucky" or read this column. In that case if you ask them to explain why their answer is correct, they probably won't be able to.

You may think that the loss would be slight by doubling 6+0 erroneously. But you have an expected gain of 29% by not doubling (Table 4) whereas by doubling it can be shown that your expectation drops to only 11%.

The exact explanation is complex; I'll present it in another column if there are several requests. The basic idea, though, is that if Player One doubles Player Two, Player Two accepts, and Player One doesn't win at once, Player Two can use the cube against Player One with great effect at Player Two's next turn.

Jacoby and Crawford discuss what is essentially the same example (they give Player Two 4+1 instead of 6+0) on pages 116-117 of their excellent *The Backgammon Book*, Viking Press, New York, 1970. Table 5 of last month's column shows that essentially the same situation occurs when Player One has 5+2 and Player Two has 4+1, 5+0, 6+0, 2+2 or 3+2 and for no other two-man end positions.

Tables 2, 3, 4 and 5 of last month's column present, for the first time anywhere, the complete, exact solutions to two-man end games in backgammon. The tables were calculated by a general method I have discovered for getting the complete exact solution to all backgammon positions that are pure races (i.e. the two sides are permanently out of contact). The intricate and difficult computer programs for computing Tables 2 through 5 were written by Don Smolen so Tables 2 through 5 are our joint work. Don was a computer scientist at Temple University. He is now trading stock options on the floor of the American Stock Exchange. A skilled backgammon player, he won the 1977 American Stock Exchange tournament.

Remember, send your questions to me care of this magazine. I won't be able to answer them individually, but I will answer some of them in this column.

TABLE 5 Doubling strategy when Player One has the move. Doubling strategy is the same, whether One has the cube or it is in the middle, except for the shaded region. If Player One has the cube he should not double for positions in the shaded region. If he makes the mistake of doubling, Two should accept. When the cube is in the middle, One should double for positions in the shaded region and Two should accept.