Oct '78

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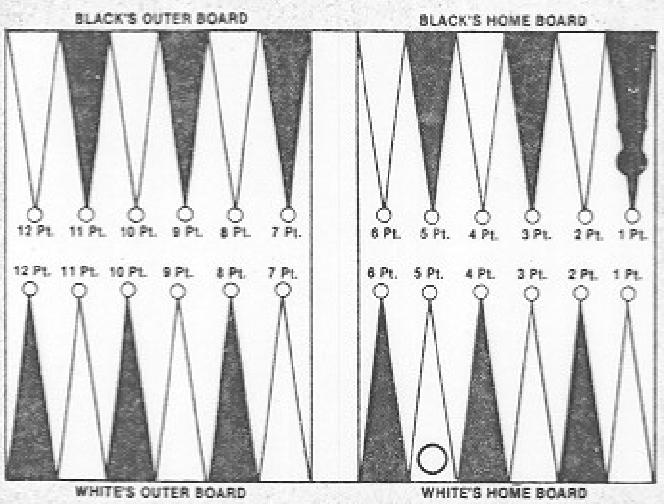
The Mathematics of Gambling

End Positions in Backgammon, Part II

by Edward Thorp

The main purpose of this month's column is to present the complete, exact solution to all backgammon positions when each player has only one or two men left in his own home board. This is the first time the solution has ever been presented. Don Smolen and I calculated it in 1975 and kept it to ourselves until now. But before we begin, I would like to comment on last month's column.

Readers of that column realize that it is often not practical or desirable to use the tables I provided during the game. Fortunately, many of these situations are covered by a handy rule that appeared in a recent "Sheinwold on Backgammon" column in the *Los Angeles Times.* Sheinwold considers the situation in Diagram 1. The problem is whether White, having rolled 6-2, should play the 2 so that he leaves his two men on 5 and 2 or on 4 and 3.



We solved this same problem last month when discussing Diagram 2 of that column. We saw then from Table 2 that leaving men on 5 and 2 is best because it gives White a 53% chance to get off on the next turn, whereas leaving men on 4 and 3 gives only a 47% chance. Now consider the general question: If you have to leave one or two men after your turn, what is the best "leave?" Assuming that the positions between which you must choose have the *same pip count,* the correct rule, which Sheinwold gives, is:

Rule for Leaving One or Two Men

(1) If possible, leave one man rather than two.
(2) If you must leave two men, leave them on different points, if possible.
(3) If you still have a choice, move off the 6-point.
(4) If you are already off the 6-point, move the man on the lower point.

It is easy to prove this rule correct by using Table 2 from last month's column. This is shown again here in condensed form as Table 1.

TABLE 1

a man on the	1 pt	2 pt	3 pt	4 pt	5 pt	6 pt
0 pt	100% 1 pip	100% 2 pips	100% 3 pips	94% 4 pips	86% 5 pips	75% 6 pips
1 pt	100% 2 pips	100% 3 pips	94% 4 pips	81% 5 pips	64% 6 pips	42% 7 pips
2 pt		72% 4 pips	69% 5 pips	64% 6 pips	53% 7 pips	36% 8 pips
3 pt			47% 6 pips	47% 7 pips	39% 8 pips	28% 9 pips
4 pt				31% 8 pips	28% 9 pips	22% 10 pips
5 pt					17% 10 pips	17% 11 pips
6 pt						11% 12 pips

To check the rule, we simply check Table 1 for each pip count to see if it always tells us which of two "leaves" to pick. For example, with a pip count of 6, part (1) of the rule says correctly that 0 pt.-6 pt. is best. Then (2) says correctly that among the three remaining two-man positions, 3 pt.-3pt. is worst. In a similar way the rule is verified in turn for positions with pip counts of 4, 5, 6, 7, 8, and 10. There's nothing to check for pip counts of 1, 2, 3, and 9 because the choices are equally good for these pip counts. There's nothing to check for counts of 11 and 12 because for these pip counts there is only one choice of position.

More examples illustrating the rule appear in *How Good are You at Backgammon: 75 Challenging Test Situations* by Nicolaos and Vassilios Tzannes, Simon and Shuster, 1974. You can use the rule to solve at once test situations 40, 41, 42, and 43. The authors give a rule (page 94) but it is neither as clear nor as simple as ours.

We proved the rule for leaving one or two men just for the case where you will have at most one more turn to play. In that case, the percentages in Table 1 let us compare two positions to see which is better. What if there is a chance that you'll have more than one turn? This could happen, for instance, if we change Diagram 1 so that Black has five men on the one point instead of four. Then Black could roll non-doubles on his next turn, leaving three men on the 1-point; White could roll 1-2 on his next turn, reducing his 5 pt.-2 pt. position to one man on the 4-point; Black could roll non-doubles again, leaving one man on the 1-point; and White then gets a second turn. It turns out that the rule gives the best choice against *all* possible positions of the opponent, not just those where you will have at most one more turn to play. I'll prove this in a later column if enough readers ask for it. (Note: There is one possible, unimportant exception that might arise, but the error is at most a small fraction of a percent.)

Now we return to the Thorp-Smolen solution of all end games with just one or two men in each home board. We will label home board positions as follows: 5+3 where there is a man on the 5-point and a man on the 3-point, with the largest number first. With both men on, say, the 4-point, we call the position 4+4. With only one man on the 5-point we write 5+0. Think of the 0 as indicating that the second man is on the 0="off" point.

There are six home board positions with one man, namely 1+0, 2+0, . . . 6+0. There are 21 home board positions with two men. Thus there are 27 one- or two- man positions for each player. (Note: In general, there are exactly $(5+r)!/5!r!$ home board positions with exactly r men. There are exactly $(6+r)!/6!r!-1$ home board positions with from one to r men. Thus, since $r=15$ is possible in the actual game, there are a total of $21!/6!\,15!-1 = 54{,}263$ different home board positions for one player. The symbol r!, read "r factorial," means $1 \times 2 \times 3 \times \ldots \times r$. Thus $1!=1$, $2!=2$, $3! = 6$, $4! = 24$, etc.)

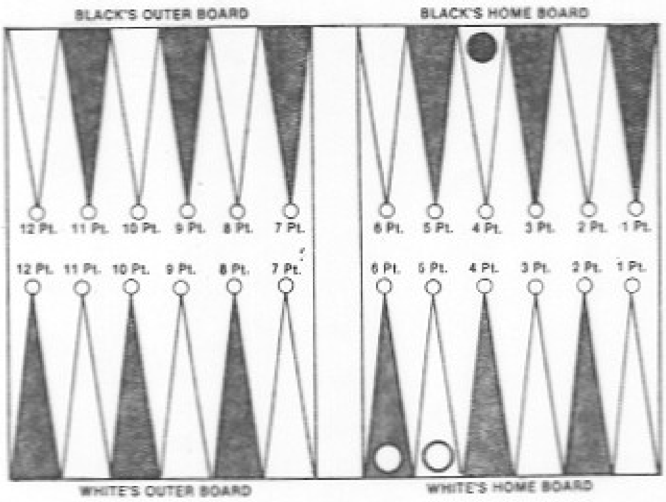
Table 2 gives the first part of our solution. It tells Player One's "expectation," rounded to the nearest percent, if One has the move and Two owns the cube. By One's expectation we mean the average number of units One can expect to win if the current stake is "one unit" and if both players follow the best strategy. Of course, if a player doesn't follow the best strategy, his opponent can expect on average to do better than Table 2 indicates.

TABLE 2

Two has → / One has ↓	A,C 6+0	2+2	3+2	4+2 5+1	5+2	3+3 4+3	6+1	5+3	6+2	4+4	6+3	5+4	6+4	5+5	6+5	6+6
2+1 A	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
3+1, 4+0	89	90	90	91	94	95	95	96	96	97	97	97	98	98	98	99
5+0	72	74	75	78	85	87	88	89	90	92	92	92	94	95	95	97
4+1	61	63	65	70	78	82	84	85	86	88	89	89	91	94	94	96
6+0	50	53	56	61	72	76	79	81	82	85	86	86	89	92	92	94
2+2	44	48	51	57	69	74	77	78	80	83	85	85	88	91	91	94
3+2	39	42	46	52	66	71	75	76	78	81	83	83	86	90	90	93
4+2, 5+1	28	32	36	44	60	66	70	72	74	78	80	80	84	68	88	92
5+2	06	11	16	27	48	55	60	63	66	71	73	73	79	84	84	89
3+3	-06	00	06	18	41	50	56	59	62	68	71	71	77	82	82	88
4+3	-06	00	06	18	41	50	56	59	61	67	70	70	76	82	82	88
6+1	-17	-10	-04	09	35	45	51	54	57	64	67	67	74	80	80	87
5+3	-22	-15	-09	05	32	41	48	51	54	61	64	64	71	78	78	85
6+2	-28	-21	-14	01	29	38	45	49	52	59	63	63	70	77	77	84
4+4	-39	-31	-23	-08	22	33	40	44	48	55	59	59	67	74	75	82
6+3	-44	-36	-28	-12	18	28	36	40	44	51	55	55	63	71	71	79
5+4	-44	-37	-30	-14	17	28	35	39	43	51	54	55	62	70	70	78
6+4	-56	-48	-40	-23	06	17	25	29	33	42	46	46	55	63	63	72
5+5	-67	-59	-51	-35	-02	10	19	24	28	37	41	42	51	59	60	70
6+5	-67	-59	-51	-36	-07	03	11	15	19	28	32	33	43	51	53	63
6+6	-78	-71	-64	-50	-25	-17	-09	-05	-01	07	12	12	23	32	36	48

The A above 6+0 means this column also applies to any count of up to 3 pips: 1+0, 2+0, 1+1, 3+0, or 2+1. The C above 6+0 means that this column also applies to 4+0, 3+1, 5+0, or 4+1. The A for Player One means the same as for Player Two.

We illustrate the use of the table with Diagram 2.



It is White's turn to move so he becomes Player One. Player Two, or Black, has the cube. We look along the row 6+5 and the column 4+3. Table 2 shows Player One's (White's) expectation as 03, so White has a 3% advantage. He expects to win on average 3% (more exactly, 2.54%) of the current stake. If the current stake is $1,000, White should accept a Black offer to "settle" the game if Black offers more than $25.40. If Black offers less, White should refuse.

Table 3 gives the expected gain or loss (to the nearest percent) for Player One when he has the move and the doubling cube is in the middle.

Unlike Table 2, in this case One has the option of doubling before he moves. If One does not double, Two will be able to double on his turn. If One doubles, Two then has the choice of accepting the double or folding. If Two accepts, play continues with doubled stakes and Two gets the cube. If Two folds, he loses the current (undoubled) stake and the game ends.

Table 4 gives the expected gain or loss for Player One when he has the move and the doubling cube. The columns for 6+4, 5+5, 6+5, and 6+6 are the same as for Table 3 so they have been omitted.

In this case, One has the option of doubling before he moves. However, in contrast to Table 3, if One does not double, he keeps the cube so Two cannot double on his next turn. If One does double, Two can accept or fold. If he accepts, the stakes are doubled, play continues, and Two gets the cube. If instead Two folds, he loses the current (undoubled) stake and the game ends. Table 5 also tells whether One should double and whether Two should accept when One has the cube.

Doubling strategy is the same whether One has the cube or it is in the middle, except for the shaded region. If he makes the mistake of doubling, Two should accept. When the cube is in the middle, One should double for positions in the shaded regions and Two should accept.

In the next month's column, we will show how to use the tables to play *perfectly* in any of the 27 x 27 = 729 end positions covered by the tables. We'll run through sample end games step by step, showing player expectation, doubling strategy, and the best way to play each roll. (Charts continue on next page.)

TABLE 3

Two has → / One has ↓	A,C 6+0	2+2	3+2	4+2 5+1	5+2	3+3 4+3	6+1	5+3	6+2	4+4	6+3	5+4	6+4	5+5	6+5	6+6
6+0 A,C	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
2+2	89	95	100	100	100	100	100	100	100	100	100	100	100	100	100	100
3+2	78	85	91	100	100	100	100	100	100	100	100	100	100	100	100	100
4+2, 5+1	56	64	72	88	100	100	100	100	100	100	100	100	100	100	100	100
5+2	11	22	32	53	95	100	100	100	100	100	100	100	100	100	100	100
3+3, 4+3	-06	01	12	36	83	100	100	100	100	100	100	100	100	100	100	100
6+1	-17	-10	-04	19	70	89	100	100	100	100	100	100	100	100	100	100
5+3	-22	-15	-09	10	63	82	95	100	100	100	100	100	100	100	100	100
6+2	-28	-21	-14	01	57	77	91	98	100	100	100	100	100	100	100	100
4+4	-39	-31	-23	-08	44	66	81	88	96	100	100	100	100	100	100	100
6+3	-44	-36	-28	-12	36	57	72	80	88	100	100	100	100	100	100	100
5+4	-44	-37	-30	-14	34	55	71	78	86	100	100	100	100	100	100	100
6+4	-56	-48	-40	-23	13	34	50	58	67	83	91	92	100	100	100	100
5+5	-67	-59	-51	-35	-01	21	39	47	56	74	83	83	100	100	100	100
6+5	-67	-59	-51	-36	-05	08	21	30	38	56	65	66	86	100	100	100
6+6	-78	-71	-64	-50	-22	-10	-02	03	07	16	24	25	46	65	71	96

TABLE 4

Two has → / One has ↓	A 2+1	3+1 4+0	5+0	4+1	6+0	2+2	3+2	4+2 5+1	5+2	3+3 4+3	6+1	5+3	6+2	4+4	6+3	5+4
6+0 A,C	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
2+2	89	89	89	89	89	95	100	100	100	100	100	100	100	100	100	100
3+2	78	78	78	78	78	85	91	100	100	100	100	100	100	100	100	100
4+2, 5+1	56	56	56	56	56	64	72	88	100	100	100	100	100	100	100	100
5+2	11	11	19	24	29	32	34	53	95	100	100	100	100	100	100	100
3+3, 4+3	-06	00	09	15	21	24	27	36	83	100	100	100	100	100	100	100
6+1	-17	-10	-00	06	13	16	19	25	70	89	100	100	100	100	100	100
5+3	-22	-15	-05	02	08	12	15	22	63	82	95	100	100	100	100	100
6+2	-28	-21	-10	-03	04	08	11	18	57	77	91	98	100	100	100	100
4+4	-39	-31	-20	-12	-04	-00	04	11	44	66	81	88	96	100	100	100
6+3	-44	-36	-24	-16	-08	-04	00	08	36	57	72	80	88	100	100	100
5+4	-44	-37	-25	-17	-09	-05	-01	07	34	55	71	78	86	100	100	100
6+4	-56	-47	-35	-26	-18	-14	-10	-01	16	34	50	58	67	83	91	92
5+5	-67	-58	-45	-36	-27	-23	-18	-10	08	21	39	47	56	74	83	83
6+5	-67	-58	-45	-37	-29	-24	-20	-12	05	14	22	30	38	56	65	66
6+6	-78	-70	-57	-49	-41	-37	-33	-25	-08	00	08	13	17	25	30	30

TABLE 5

Two has → / One has ↓	A 2+1	3+1 4+0	4+1 5+0	6+0	2+2	3+2	4+2 5+1	5+2	3+3 4+3	6+1	5+3	6+2	4+4	5+4 6+3	6+4	5+5	6+5 6+6
2+1 A																	
3+1, 4+0						One Should Double											
4+1, 5+0						Two Should Fold											
6+0	Two May Accept or Fold																
2+2																	
3+2	One Should Double																
4+2, 5+1	Two Should Acept																
5+2																	
3+3																	
4+3																	
6+1																	
5+3	One Should Not Double																
6+2	Two Should Not Accept																
4+4																	
5+4, 6+3																	
6+4																	
5+5																	
6+5																	
6+6																	