The Mathematics of Gambling: End Positions in Backgammon

by Edward Thorp

This is the start of a regular column about mathematical aspects of gambling and games. I hope to write so that if you skip the mathematical parts you will still understand the columns and find them useful. Future columns will include topics such as: Winning Gambling Systems; Games that Do Not Have Winning Systems; Counting All the Cards in Blackjack; Those Slick Bridge Columns are Sometimes Wrong; When to Insure at Blackjack to Reduce Risk; Optimal Strategies in Backgammon.

I welcome questions and comments and will respond briefly in this column to correspondence that is of particular interest to readers.

Today's column, End Positions in Backgammon, is the first of several on the game. In it you learn useful but simple odds for bearing off with only two men left. Most good players already know this. But, good players, don't go away. In later columns you will learn facts about backgammon that no one in the world has ever known before.

As an introduction to end positions, suppose you are White and it is your turn to roll in the position of Diagram 1. The doubling cube is in the middle.

Questions:

1. What is your chance to win?

2. Should you double?

3. How much do you gain or lose by doubling?

4. If you double, should Black accept?

5. How much does Black gain or lose by accepting your double?

White wins only if he bears off on his next roll. So to help us solve end positions of this type, we calculate a table of chances to take off two men in one roll. The exact result is given in Table 1, and the chances to the nearest percent are given by Table 2.

To illustrate the use of Table 1, suppose you have a man on the five point and a man on the two point. Table 1 gives 19 chances in 36 to take both men off on the next roll.

This means the exact chance you win in Diagram 1 is 19/36 = 0.5277... . Table 2 gives this to the nearest percent as 53%. This answers question 1.

To see how Table 1 is calculated, recall that there are 36 equally likely outcomes for the roll of two dice. These are listed in Table 3. Think of the two dice as labelled "first" and "second".

It might help to use a red die for the "first" die and a white one for the "second" die. Then if the red (first) die shows 5 and the white (second) die shows 2, we call the outcome 5-2. If instead the first die shows 2 and the second die shows 5, this is a different one of the 36 rolls and we call it 2-5. Outcomes are named x-y where x is the number the first die shows and y is the number the second die shows.

To see that white has 19 chances in 36 to win, we simply count winning rolls in Table 3. If either die shows at least 2 and the other shows at least 5, white wins. He also wins with 2-2, 3-3 and 4-4. This gives the 19 (shaded) winning outcomes in Table 3.

As another example, suppose the two men to bear off are both on the six point. Then if the two dice are different, white can't come off in one turn. Of the six doubles, only 3-3 or higher works. This gives 4 ways in 36 or 11%, in agreement with Tables 2 and 3. This simple counting method produces all the numbers in Table 1.

Now we are ready to answer question 2: Should white double, in Diagram 1? The answer is Yes, and here's why. We have seen that white wins on average 19 times in 36. If we call the stake 1 unit, then if he does not double, in 36 times he wins 1 unit 19 times and loses 1 unit 17 times for a gain of 2 units/36 times = 1/18 = 0.055... . If white does double, black can either accept or fold. Suppose black accepts. Then the stakes are 2 units and a calculation like the previous one shows white gains an average of 4 units/36 times = 1/9 = 0.111... unit per time. White gains twice as much as if he did not double. If instead black folds, then white wins 1 unit at once, which is even better.

This also answers the rest of the questions. Answer to question 3: white gains an extra 5.55% of a unit, on average, by doubling. Answer to question 4: black should accept. He loses 1/9 unit on average by accepting and 1 unit for sure by folding. This answers question 5: if he makes the error of folding, he loses an extra 8/9 unit or 89%.

The usefulness of Table 2 is generally limited to situations where you have just one or two rolls left before the game ends. But it is surprising how often the table is valuable. Here are some more examples to help alert you to these situations. In Diagram 2, Black has the doubling cube. White has just rolled 2-1. How does he play it? If black rolls doubles on the next turn, he wins at once and it won't matter what white did. So white only needs to consider the case where black does not roll doubles. Then white will have one more turn and he wants to leave himself with the greatest chance to bear off on that turn. White can move one man from the 5 point to the 4 point and one man from the 5 point to the 3 point. By Table 2, this gives him a 47% chance to win if black does not roll doubles. Or, white can move one man from the 5 point to the 2 point, leaving the other man on the 5 point. This gives him a 53% chance to win if black does not roll doubles, so this is the best way to play the 2-1.

In Diagram 3, white's problem is to avoid a backgammon: if black wins before the white men escape from black's home board, black will win 3 units. Otherwise he will only gammon white for two units. White must use the 4 to move the man on the bar to the black 4 point (dotted circle). White can move this man on to the black 5 point in which case, if black does not roll doubles, white's situation on his last turn is shown in Diagram 3a. The chance for white to remove both men from black's home board on the next roll is the same as the chance to bear off both men when one is on the 4 point and the other is on the 2 point. By Table 2 this is 64%.

Suppose instead white plays both men to the black 4 point.

Then Diagram 3b shows the board if he survives black's next roll. His chance to save himself from backgammon is the same as bearing off two men from the 3 point in one roll. Table 2 gives 47%. Therefore the play in Diagram 3a is best.

If instead white rolled 4-2 in Diagram 3, he could play to leave his two back men on the black 2 and 7 points, giving an 86% chance (Table 2, man on 5 point and man on 0 point) to escape black's home board on the next roll. Or he could play to leave his two back men on the black 5 and 4 points. This gives only a 69% chance so is inferior.

An outstanding book on backgammon is "Backgammon" by Paul Magriel, The New York Times Publishing Company, 1977, $20. Most of Table 1 appears there on page 404. A handy reference for practical play is the "Backgammon Calculator", Doubleday, 1974, $1.95. This handy cardboard wheel has most of Table 2 on the back.

Here are some questions to check your understanding of today's column. Refer to Diagram 2.

1. Should black double, after white makes the best move?
2. How much would black gain or lose by so doubling?

3. Should white accept a black double? If he does, instead of folding, how much does he gain or lose?

4. What is the best way for white to play 3-2 in Diagram 2?

Next month this column will present, for the first time anywhere, the complete analysis and best strategies for all end games with just one or two men left in each home board.

TABLE 1

Chances out of 36 to bear off in one roll
with one or two men left.

a man on the	0 pt	1 pt	2 pt	3 pt	4 pt	5 pt	6 pt
0 pt	off	36	36	36	34	31	27
1 pt	36	36	36	34	29	23	15
2 pt	36	36	26	25	23	19	13
3 pt	36	34	25	17	17	14	10
4 pt	34	29	23	17	11	10	8
5 pt	31	23	19	14	10	6	6
6 pt	27	15	13	10	8	6	4

TABLE 2

Percentage chances to bear off in one roll with one or two men left.

a man on the	0 pt	1 pt	2 pt	3 pt	4 pt	5 pt	6 pt
0 pt	off	100%	100%	100%	94%	86%	75%
1 pt	100%	100%	100%	94%	81%	64%	42%
2 pt	100%	100%	72%	69%	64%	53%	36%
3 pt	100%	94%	69%	47%	47%	39%	28%
4 pt	94%	81%	64%	47%	31%	28%	22%
5 pt	86%	64%	53%	39%	28%	17%	17%
6 pt	75%	42%	36%	28%	22%	17%	11%

TABLE 3

The 36 equally likely outcomes of the roll of two dice.

second die shows→ ↓ first die shows	1	2	3	4	5	6
1	1-1	1-2	1-3	1-4	1-5	1-6
2	2-1	2-2	2-3	2-4	2-5	2-6
3	3-1	3-2	3-3	3-4	3-5	3-6
4	4-1	4-2	4-3	4-4	4-5	4-6
5	5-1	5-2	5-3	5-4	5-5	5-6
6	6-1	6-2	6-3	6-4	6-5	6-6

DIAGRAM 1

You are White, it is your turn to roll, and the doubling cube is in the middle.

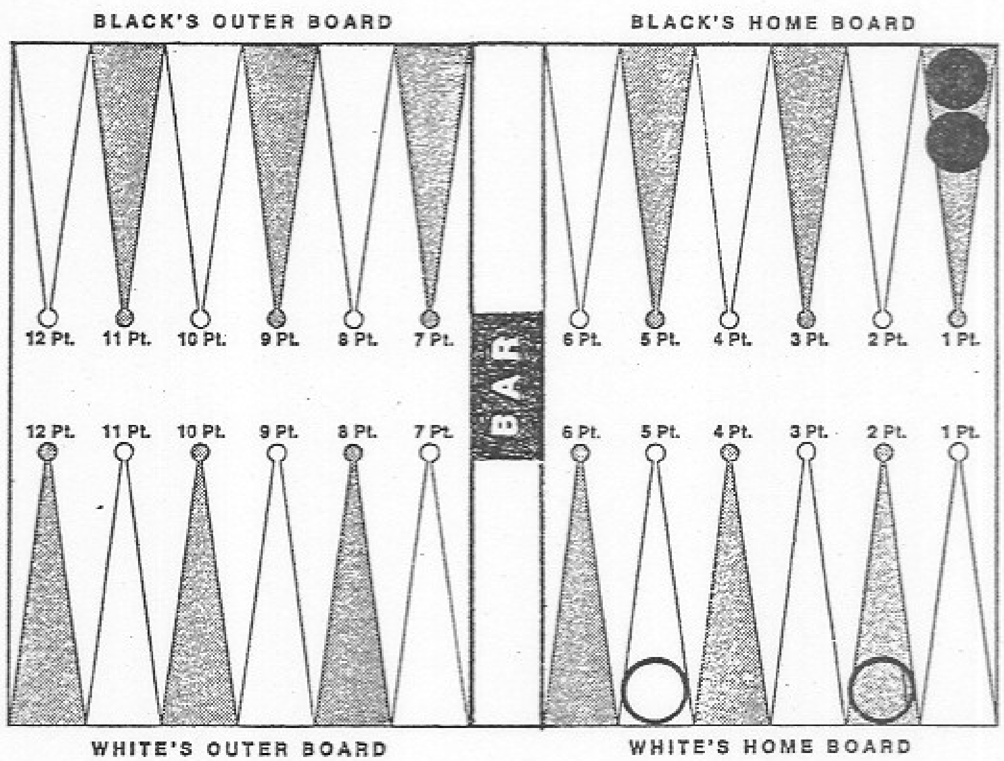

DIAGRAM 2

Black has the doubling cube. White has just rolled 2-1. What is the best move?

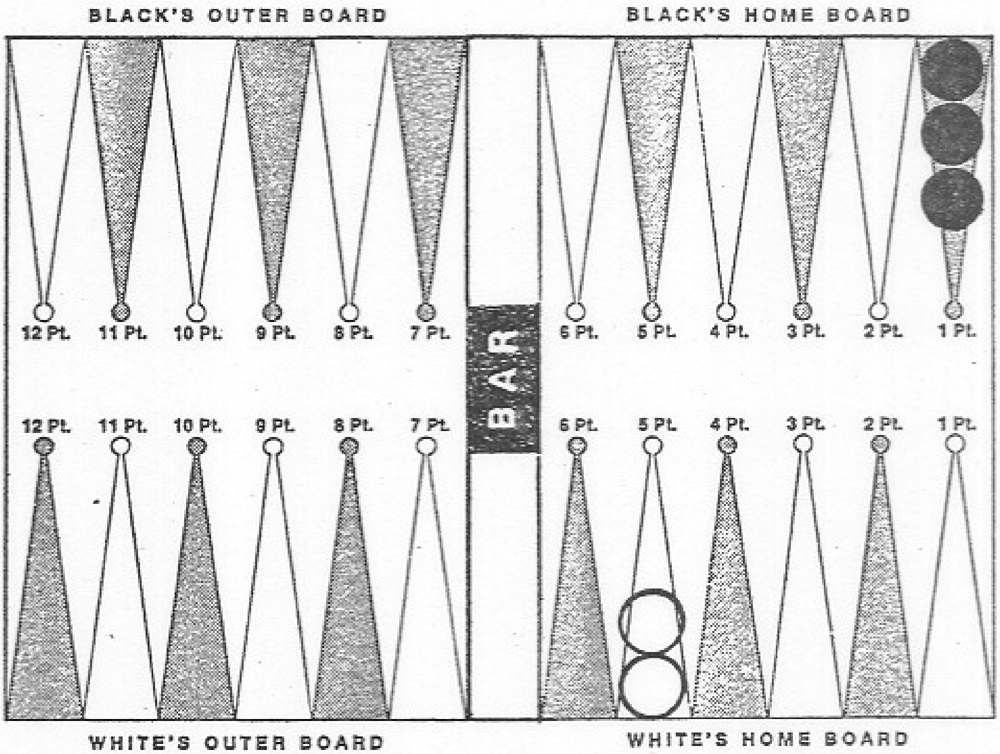

DIAGRAM 3

White has just rolled 4-1. What is the best move?

steps from Black outer board: 0 1 2 3 4 5 6

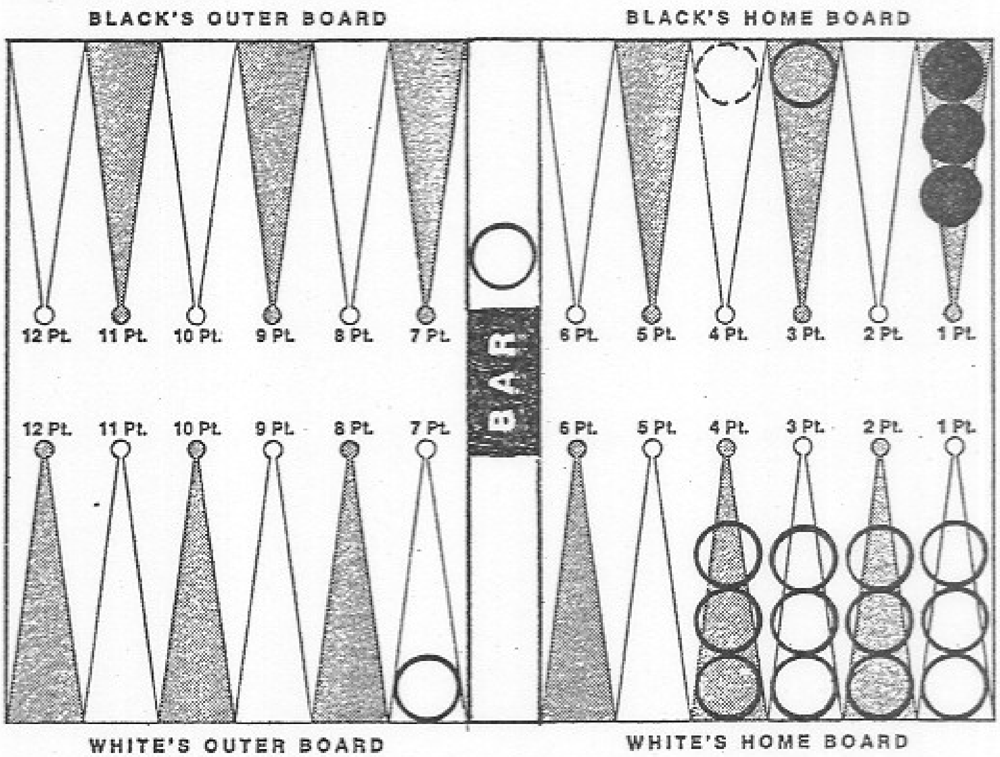

DIAGRAM 3a

After White plays from bar to the Black 5 point and Black does not roll doubles.

steps from Black outer board: 0 1 2 3 4 5 6

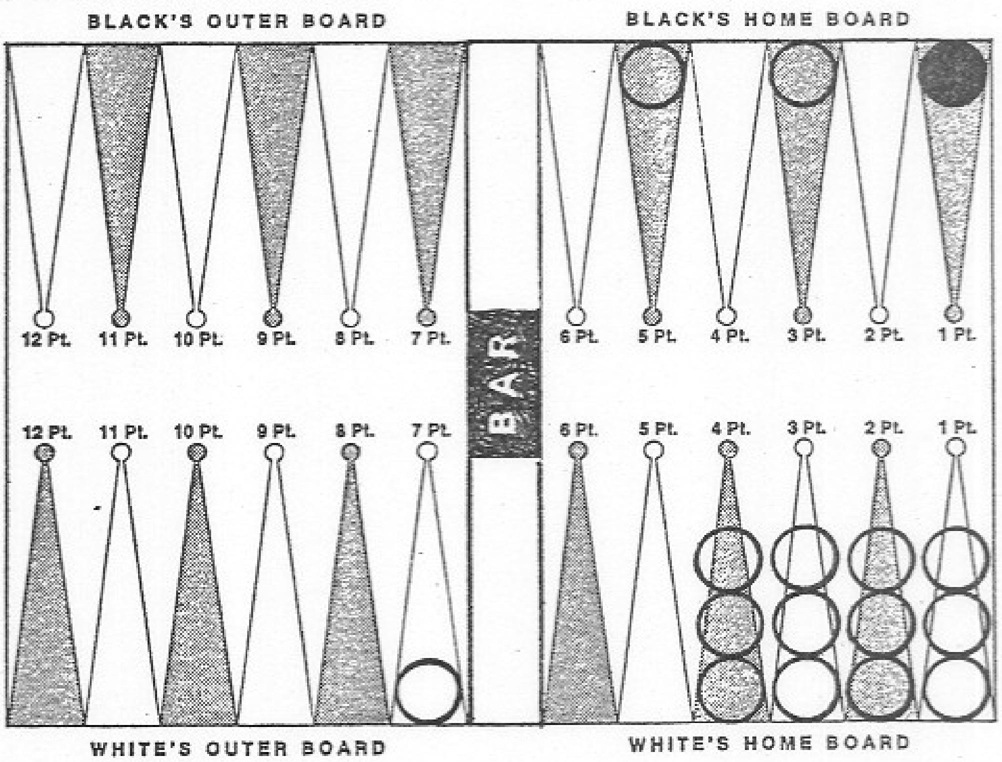

DIAGRAM 3b

After White plays both men to the Black 4 point and Black does not roll doubles.

steps from Black outer board: 0 1 2 3 4 5 6