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Editor

Computer Games I

With 97 Illustrations

Springer-Verlag
New York Berlin Heidelberg
London Paris Tokyo

1.3. End Positions in Backgammon

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Originally published in: *Gambling Times*, October 1978, November 1978, December 1978.

PART I

In "End Positions in Backgammon," you learn useful but simple odds for bearing off with only two men left. Most good players already know this. But, good players, do not go away. Later you will learn facts about backgammon that no one in the world has ever known before. As an introduction to end positions, suppose you are White and it is your turn to roll in the position of Figure 1. The doubling cube is in the middle.

Questions:

1. What is your chance to win?
2. Should you double?
3. How much do you gain or lose by doubling?
4. If you double, should Black accept?
5. How much does Black gain or lose by accepting your double?

White wins only if he bears off on his next roll. So to help us solve end positions of this type, we calculate a table of chances to take off two men in one roll. The exact result is given in Table 1, and the chances to the nearest percent are given by Table 2. To illustrate the use of Table 1, suppose you have a man on the 5 point and a man on the 2 point. Table 1 gives 19 chances in 36 to take both men off on the next roll. This means the exact chance you win in Figure 1 is 19/36 = 0.5277.... Table 2 gives this to the nearest percent as 53%. This answers Question 1.

To see how Table 1 is calculated, recall that there are 36 *equally likely* outcomes for the roll of two dice. These are listed in Table 3. Think of the two dice as labeled "first" and "second". It might help to use a red die for the "first" die and a white one for the "second" die. Then if the red (first) die shows 5 and

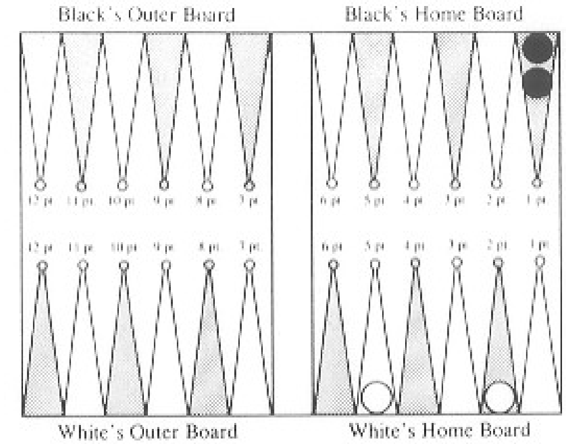


Figure 1. You are White, it is your turn to roll, and the doubling cube is in the middle.

Table 1. Chances out of 36 to bear off in one roll with one or two men left.

A man on the	0 pt.	1 pt.	2 pt.	3 pt.	4 pt.	5 pt.	6 pt.
0 pt.	off	36	36	36	34	31	27
1 pt.	36	36	36	34	29	23	15
2 pt.	36	36	26	25	23	19	13
3 pt.	36	34	25	17	17	14	10
4 pt.	34	29	23	17	11	10	8
5 pt.	31	23	19	14	10	6	6
6 pt.	27	15	13	10	8	6	4

Table 2. Percentage chances to bear off in one roll with one or two men left.

A man on the	0 pt.	1 pt.	2 pt.	3 pt.	4 pt.	5 pt.	6 pt.
0 pt.	off	100%	100%	100%	94%	86%	75%
1 pt.	100%	100%	100%	94%	81%	64%	42%
2 pt.	100%	100%	72%	69%	64%	53%	36%
3 pt.	100%	94%	69%	47%	47%	39%	28%
4 pt.	94%	81%	64%	47%	31%	28%	22%
5 pt.	86%	64%	53%	39%	28%	17%	17%
6 pt.	75%	42%	36%	28%	22%	17%	11%

Table 3. The 36 equally likely outcomes of the roll of two dice.

Second die shows→ ↓First die shows	1	2	3	4	5	6
1	1–1	1–2	1–3	1–4	1–5	1–6
2	2–1	2–2	2–3	2–4	2–5	2–6
3	3–1	3–2	3–3	3–4	3–5	3–6
4	4–1	4–2	4–3	4–4	4–5	4–6
5	5–1	5–2	5–3	5–4	5–5	5–6
6	6–1	6–2	6–3	6–4	6–5	6–6

the white (second) die shows 2, we call the outcome 5–2. If instead the first die shows 2 and the second die shows 5, this is a different one of the 36 rolls and we call it 2–5. Outcomes are named x–y where x is the number the first die shows and y is the number the second die shows.

To see that White has 19 chances in 36 to win, we simply count winning rolls in Table 3. If either die shows at least 2 and the other shows at least 5, White wins. He also wins with 2–2, 3–3, and 4–4. This gives the 19 (shaded) winning outcomes in Table 3. As another example, suppose the two men to bear off are both on the 6 point. Then if the two dice are different, White can not come off in one turn. Of the six doubles, only 3–3 or higher works. This gives 4 ways in 36 or 11%, in agreement with Tables 2 and 3. This simple counting method produces all the numbers in Table 1.

Now we are ready to answer Question 2: Should White double in Figure 1? The answer is Yes, and here's why. We have seen that White wins on average 19 times in 36. If we call the stake 1 unit then, if he does not double, in 36 times he wins 1 unit 19 times and loses 1 unit 17 times for a gain of 2 units/36 times = 1/18 = 0.055.... If White does double, Black can either accept or fold. Suppose Black accepts. Then the stakes are 2 units and a calculation like the previous one shows White gains an average of 4 units/36 times = 1/9 = 0.111 ... unit per time. White gains twice as much as if he did not double. If instead Black folds, then White wins 1 unit at once, which is even better.

This also answers the rest of the questions. Answer to Question 3: White gains an extra 5.55% of a unit, on average, by doubling. Answer to Question 4: Black should accept. He loses 1/9 unit on average by accepting and 1 unit for sure by folding. This answers Question 5: if he makes the error of folding, he loses an extra 8/9 unit or 89%.

The usefulness of Table 2 is generally limited to situations where you have just one or two rolls left before the game ends. But it is surprising how often the table is valuable. Here are some more examples to help alert you to these situations. In Figure 2, Black has the doubling cube. White has just rolled

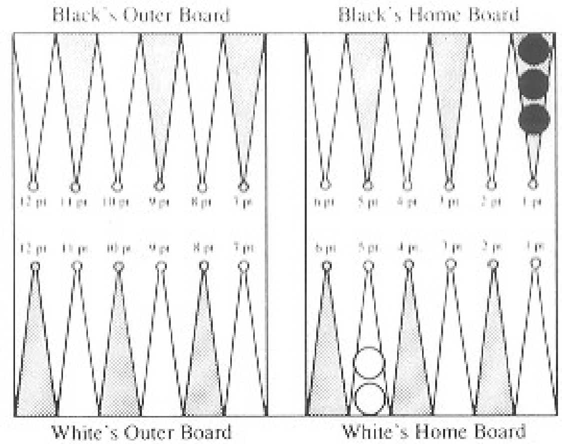


Figure 2. Black has the doubling cube. White has just rolled 2–1. What is the best move?

2–1. How does he play it? If Black rolls doubles on the next turn, he wins at once and it won't matter what White did. So White only needs to consider the case where Black does not roll doubles. Then White will have one more turn and he wants to leave himself with the greatest chance to bear off on that turn. White can move one man from the 5 point to the 4 point and one man from the 5 point to the 3 point. By Table 2, this gives him a 47% chance to win if Black does not roll doubles. Or, White can move one man from the 5 point to the 2 point, leaving the other man on the 5 point. This gives him a 53% chance to win if Black does not roll doubles, so this is the best way to play the 2–1.

In Figure 3, White's problem is to avoid a backgammon: if Black wins before

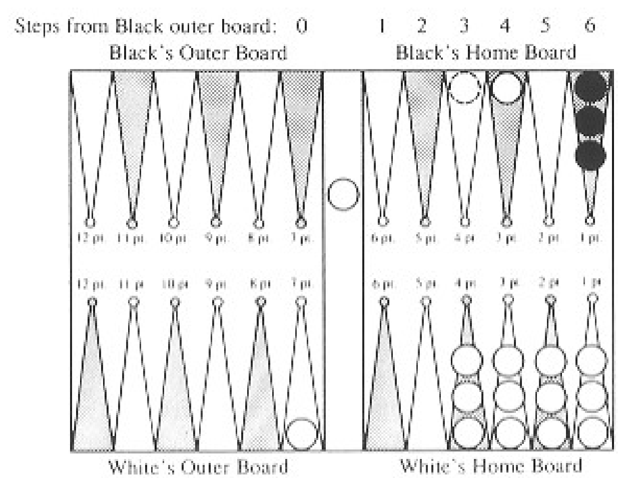


Figure 3. White has just rolled 4–1. What is the best move?

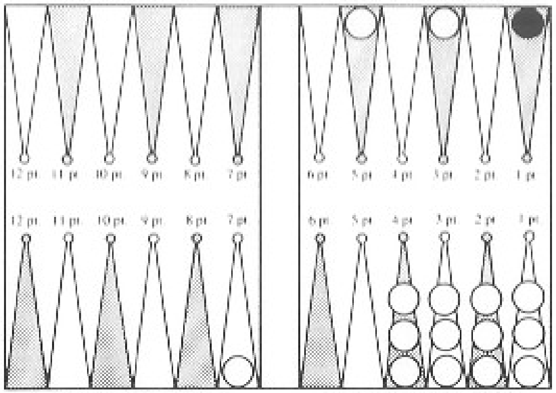


Figure 3(a). After White plays from bar to the black 5 point and Black does not roll doubles.

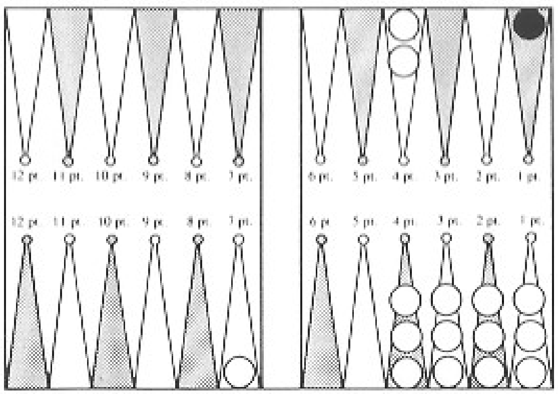


Figure 3(b). After White plays both men to the black 4 point and Black does not roll doubles.

the white men escape from Black's home board, Black will win 3 units. Otherwise he will only gammon White for two units. White must use the 4 to move the man on the bar to the black 4 point (dotted circle). White can move this man on to the black 5 point in which case, if Black does not roll doubles, White's situation on his last turn is shown in Figure 3(a). The chance for White to remove both men from Black's home board on the next roll is the same as the chance to bear off both men when one is on the 4 point and the other is on the 2 point. By Table 2 this is 64%.

Suppose instead White plays both men to the black 4 point. Then Figure 3(b) shows the board if he survives Black's next roll. His chance to save himself

from backgammon is the same as bearing off two men from the 3 point in one roll. Table 2 gives 47%. Therefore the play in Figure 3(a) is best. If instead White rolled 4–2 in Figure 3, he could play to leave his two back men on the black 2 and 7 points, giving an 86% chance (Table 2, man on 5 point and man on 0 point) to escape Black's home board on the next roll. Or he could play to leave his two back men no the black 5 and 4 points. This gives only a 69% chance, so is inferior.

An outstanding book on backgammon is Magriel (1977). Most of Table 1 appears there on page 404. A handy reference for practical play is the Doubleday Device (1974). This handy cardboard wheel has most of Table 2 on the back.

Here are some questions to check your understanding of what has been discussed so far. Refer to Figure 2.

1. Should Black double, after White makes the best move?
2. How much would Black gain or lose by so doubling?
3. Should White accept a Black double? If he does, instead of folding, how much does he gain or lose?
4. What is the best way for White to play 3–2 in Figure 2?

PART II

Next we consider the complete exact solution to all backgammon positions when each player has only one or two men left in his own home board. This is the first time this has ever been presented. It was calculated in 1975 by Don Smolen and myself and kept to ourselves until now. However, first I have a comment on Part I of this article.

Readers of Part I realize that it is often not practical or desirable to use the tables during the game. Fortunately, many of these situations are covered by a handy rule that appeared, for instance, in a recent "Sheinwold on Backgammon" column from the *Los Angeles Times*. Sheinwold considers the situation in Figure 4. The problem is whether White should play the 2 so that he leaves his two men on 5 and 2 or on 4 and 3.

We solved this same problem when discussing Figure 2. We saw then from Table 2 that leaving men on 5 and 2 is best because it gives White a 53% chance to get off on the next turn, whereas leaving men on 4 and 3 gives only a 47% chance. Now consider the general question: If you have to leave one or two men after your turn, what is the best "leave"? Assuming that the positions between which you must choose have the *same pip count*, the correct rule, which Sheinwold gives, is:

Rule for leaving one or two men.

(1) If possible, leave one man rather than two.
(2) If you must leave two men, leave them on different points, if possible.
(3) If you still have a choice, move off the 6 point.
(4) If you are already off the 6 point, move the man on the lower point.

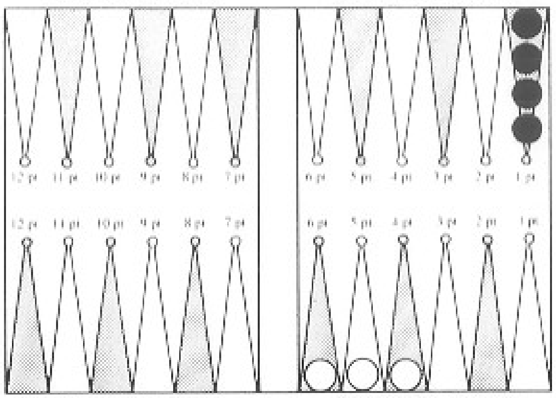


Figure 4. White rolls 6–2 and must bear a man off the 6 point. How should he play the 2?

It is easy to prove this rule is correct by using Table 2. This is shown again here in condensed form as Table 4. To check the rule, we simply check Table 4 for each pip count to see if it always tells us which of two "leaves" to pick. For example, with a pip count of 6, part (1) of the rule says correctly that 0 pt.–6 pt. is best. Then (2) says correctly that among the three remaining

Table 4. Percentage chances to bear off in one roll with one or two men left (from Table 2).

A man on the	1 pt.	2 pt.	3 pt.	4 pt.	5 pt.	6 pt.
0 pt.	100% 1 pip	100% 2 pips	100% 3 pips	94% 4 pips	86% 5 pips	75% 6 pips
1 pt.	100% 2 pips	100% 3 pips	94% 4 pips	81% 5 pips	64% 6 pips	42% 7 pips
2 pt.		72% 4 pips	69% 5 pips	64% 6 pips	53% 7 pips	36% 8 pips
3 pt.			47% 6 pips	47% 7 pips	39% 8 pips	28% 9 pips
4 pt.				31% 8 pips	28% 9 pips	22% 10 pips
5 pt.					17% 10 pips	17% 11 pips
6 pt.						11% 12 pips

two-man positions, 3 pt.–3 pt. is worst. In a similar way, the rule is verified in turn for positions with pip counts of 4, 5, 6, 7, 8, and 10. There is nothing to check for pip counts of 1, 2, 3, and 9 because the choices are equally good for these pip counts. There is nothing to check for counts of 11 and 12 because for these pip counts there is only one choice of position.

More examples illustrating the rule appear in Tzannes and Tzannes (1974). You can use the rule to solve at once test situations 40, 41, 42, and 43. The authors give a rule (page 94) but it is neither as clear nor as simple as ours.

We proved the rule for leaving one or two men just for the case where you will have at most one more turn to play. In that case, the percentages in Table 4 let us compare two positions to see which is better. What if there is a chance that you will have more than one turn? This could happen, for instance, if we change Figure 4 so that Black has five men on the 1 point instead of four. Then Black could roll nondoubles on his next turn, leaving three men on the 1 point; White could roll 1–2 on his next turn, reducing his 5 pt.–2. pt. position to one man on the 4 point; Black could roll nondoubles again, leaving one man on the 1 point; and White then gets a second turn. It turns out that the rule gives the best choice against all possible positions of the opponent, not just those where you will have at most one more turn to play. (Note: There is one possible unimportant exception that might arise, but the error is at most a small fraction of a percent.)

Now we return to the Thorp–Smolen solution of all endgames with just one or two men in each home board. We will label home board positions as follows: 5+3 where there is a man on the 5 point and a man on the 3 point, with the largest number first. With both men on say the 4 point, we call the position 4+4. With only one man on say the 5 point we write 5+0. Think of the 0 as indicating that the second man is on the 0 = Off point.

There are six home-board positions with one man, namely 1+0, 2+0, ..., 6+0. There are 21 home-board positions with two men. Thus there are 27 one- or two-man positions for each player. (Note: In general, there are exactly $(5 + r)!/5!\,r!$ home-board positions with exactly r men. There are exactly $(6 + r)!/6!\,r! - 1$ home-board positions with from one to r men. Thus since $r = 15$ is possible in the actual game, there are a total of $21!/6!\,15! - 1 = 54{,}263$ different home-board positions for one player. The symbol $r!$, read "r factorial", means $1 \times 2 \times 3 \times \cdots \times r$. Thus $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, etc.)

Table 5 gives the first part of our solution. It tells Player I's "expectation" if I has the move and II owns the cube. By I's expectation we mean the average number of units I can expect to win if the current stake is "one unit," and if both players follow the best strategy. Of course, if a player does not follow the best strategy, his opponent can expect, on average, to do better than Table 5 indicates.

The A above 6+0 means this column also applies to any count of up to three pips: 1+0, 2+0, 1+1, 3+0, or 2+1. The C above 6+0 means that this column also applies to 4+0, 3+1, 5+0, or 4+1. The A for Player I means

Table 5. Player I's expected gain or loss, rounded to the nearest percent when Player I is to move and Player II owns the doubling cube. Column labels refer to the II home-board position and row labels refer to the I home-board position. Letters A and C (see text) indicate other Player I or Player II positions that are included with the given headings.

II has→ / I has↓	A, C 6+0	2+2	3+2	4+2 5+1	5+2	3+3 4+3	6+1	5+3	6+2	4+4	6+3	5+4	6+4	5+5	6+5	6+6
2+1 A	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
3+1, 4+0	89	90	90	91	94	95	95	96	96	97	97	97	98	98	98	99
5+0	72	74	75	78	85	87	88	89	90	92	92	92	94	95	95	97
4+1	61	63	65	70	78	82	84	85	86	88	89	89	91	94	94	96
6+0	50	53	56	61	72	76	79	81	82	85	86	86	89	92	92	94
2+2	44	48	51	57	69	74	77	78	80	83	85	85	88	91	91	94
3+2	39	42	46	52	66	71	75	76	78	81	83	83	86	90	90	93
4+2, 5+1	28	32	36	44	60	66	70	72	74	78	80	80	84	88	88	92
5+2	06	11	16	27	48	55	60	63	66	71	73	73	79	84	84	89
3+3	−06	00	06	18	41	50	56	59	62	68	71	71	77	82	82	88
4+3	−06	00	06	18	41	50	56	59	61	67	70	70	76	82	82	88
6+1	−17	−10	−04	09	35	45	51	54	57	64	67	67	74	80	80	87
5+3	−22	−15	−09	05	32	41	48	51	54	61	64	64	71	78	78	85
6+2	−28	−21	−14	01	29	38	45	49	52	59	63	63	70	77	77	84
4+4	−39	−31	−23	−08	22	33	40	44	48	55	59	59	67	74	75	82
6+3	−44	−36	−28	−12	18	28	36	40	44	51	55	55	63	71	71	79
5+4	−44	−37	−30	−14	17	28	35	39	43	51	54	55	62	70	70	78
6+4	−56	−48	−40	−23	06	17	25	29	33	42	46	46	55	63	63	72
5+5	−67	−59	−51	−35	−02	10	19	24	28	37	41	42	51	59	60	70
6+5	−67	−59	−51	−36	−07	03	11	15	19	28	32	33	43	51	53	63
6+6	−78	−71	−64	−50	−25	−17	−09	−05	−01	07	12	12	23	32	36	48

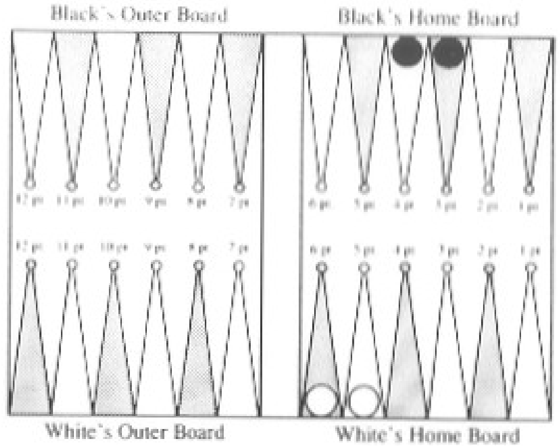


Figure 5. It is White's turn. Black has the cube. Who has the advantage? How much?

the same as for Player II. We illustrate the use of the table with Figure 5. It is White's turn to move so he becomes Player I. We look along the row 6+5 and the column 4+3. Table 5 shows Player I's (White's) expectation as 03 so White has a 3% advantage. He expects to win on average 3% (more exactly 2.54%) of the current stake. If the current stake is $1,000, White should accept a Black offer to "settle" the game if Black offers more than $25.40. If Black offers less, White should refuse.

Table 6 gives the expected gain or loss for Player I when he has the move and the doubling cube is in the middle. Unlike Table 5, in this case I has the option of doubling before he moves. If I does not double, II will be able to double on his next turn. If I doubles, II then has the choice of accepting the double or folding. If II accepts, play continues with doubled stakes and II gets the cube. If II folds, he loses the current (undoubled) stake and the game ends. Table 8 tells whether I should double and whether II should accept.

Table 7 gives the expected gain or loss for Player I when he has the move and the doubling cube. In this case, I has the option of doubling before he moves. However, in contrast to Table 6, if I does not double he keeps the cube so II cannot double on his next turn. If I does double, II can accept or fold. If he accepts, the stakes are doubled, play continues, and II gets the cube. If instead II folds, he loses the current (undoubled) stake and the game ends. Table 8 also tells whether I should double and whether II should accept when I has the cube.

In Part III, we will show how to use the tables to play *perfectly* in any of the 27 × 27 = 729 end positions covered by the tables. We will run through sample endgames step by step, showing player expectation, doubling strategy, and the best way to play each roll.

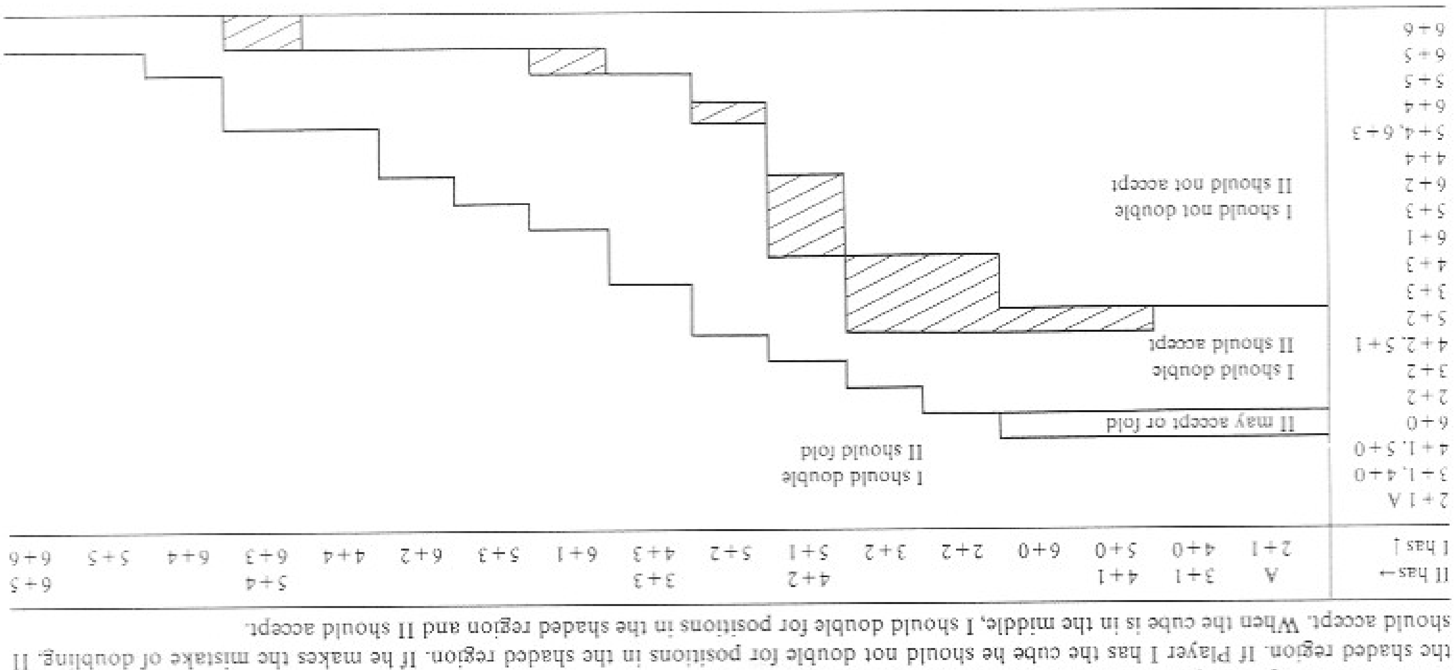
Table 6. Player I's expected gain or loss, rounded to the nearest percent, when Player I is to move and the doubling cube is in the middle. The column labels refer to the II home board and the row labels refer to the I home board. Headings are interpreted as in Table 2.

II has→ I has↓	A, C 6+0	2+2	3+2	4+2 5+1	5+2	3+3 4+3	6+1	5+3	6+2	4+4	6+3	5+4	6+4	5+5	6+5	6+6
6+0 A, C	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
2+2	89	95	100	100	100	100	100	100	100	100	100	100	100	100	100	100
3+2	78	85	91	100	100	100	100	100	100	100	100	100	100	100	100	100
4+2, 5+1	56	64	72	88	100	100	100	100	100	100	100	100	100	100	100	100
5+2	11	22	32	53	95	100	100	100	100	100	100	100	100	100	100	100
3+3, 4+3	−06	01	12	36	83	100	100	100	100	100	100	100	100	100	100	100
6+1	−17	−10	−04	19	70	89	100	100	100	100	100	100	100	100	100	100
5+3	−22	−15	−09	10	63	82	95	100	100	100	100	100	100	100	100	100
6+2	−28	−21	−14	01	57	77	91	98	100	100	100	100	100	100	100	100
4+4	−39	−31	−23	−08	44	66	81	88	96	100	100	100	100	100	100	100
6+3	−44	−36	−28	−12	36	57	72	80	88	100	100	100	100	100	100	100
5+4	−44	−37	−30	−14	34	55	71	78	86	100	100	100	100	100	100	100
6+4	−56	−48	−40	−23	13	34	50	58	67	83	91	92	100	100	100	100
5+5	−67	−59	−51	−35	−01	21	39	47	56	74	83	83	100	100	100	100
6+5	−67	−59	−51	−36	−05	08	21	30	38	56	65	66	86	100	100	100
6+6	−78	−71	−64	−50	−22	−10	−02	03	07	16	24	25	46	65	71	96

Table 7. Player I's expected gain or loss, rounded to the nearest percent when Player I is to move and also has the doubling cube. Headings are interpreted as in Table 2. The columns for 6+4, 5+5, 6+5, and 6+6 are the same as for Table 3 so they have been omitted.

II has→ I has↓	A 2+1	3+1 4+0	5+0	4+1	6+0	2+2	3+2	4+2 5+1	5+2	3+3 4+3	6+1	5+3	6+2	4+4	6+3	5+4
6+0 A, C	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
2+2	89	89	89	89	89	95	100	100	100	100	100	100	100	100	100	100
3+2	78	78	78	78	78	85	91	100	100	100	100	100	100	100	100	100
4+2, 5+1	56	56	56	56	56	64	72	88	100	100	100	100	100	100	100	100
5+2	11	11	19	24	29	32	34	53	95	100	100	100	100	100	100	100
3+3, 4+3	−06	00	09	15	21	24	27	36	83	100	100	100	100	100	100	100
6+1	−17	−10	−00	06	13	16	19	25	70	89	100	100	100	100	100	100
5+3	−22	−15	−05	02	08	12	15	22	63	82	95	100	100	100	100	100
6+2	−28	−21	−10	−03	04	08	11	18	57	77	91	98	100	100	100	100
4+4	−39	−31	−20	−12	−04	−00	04	11	44	66	81	88	96	100	100	100
6+3	−44	−36	−24	−16	−08	−04	00	08	36	57	72	80	88	100	100	100
5+4	−44	−37	−25	−17	−09	−05	−01	07	34	55	71	78	86	100	100	100
6+4	−56	−47	−35	−26	−18	−14	−10	−01	16	34	50	58	67	83	91	92
5+5	−67	−58	−45	−36	−27	−23	−18	−10	08	21	39	47	56	74	83	83
6+5	−67	−58	−45	−37	−29	−24	−20	−12	05	14	22	30	38	56	65	66
6+6	−78	−70	−57	−49	−41	−37	−33	−25	−08	00	08	13	17	25	30	30

Table 8. Doubling strategy when Player I has the move. Doubling strategy is the same, whether I has the cube or it is in the middle, except for the shaded region. If Player I has the cube he should not double for positions in the shaded region. If he makes the mistake of doubling, II should accept. When the cube is in the middle, I should double for positions in the shaded region and II should accept.



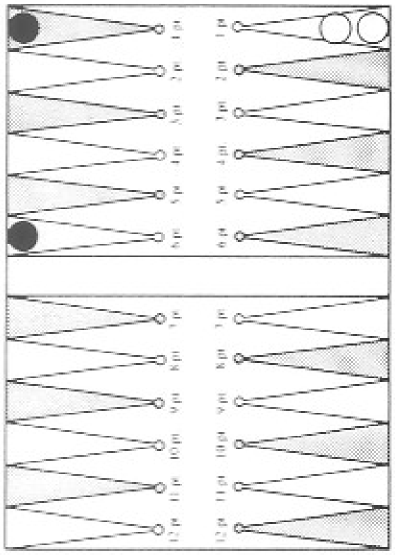


Figure 6. (Tzannes and Tzannes) Black is to move. The cube is in the middle. He doubles. Is this correct? Should White accept?

PART III

Here we will illustrate and explain the use of the tables presented in Part II. Consider first Situation 74 from Tzannes and Tzannes (1974). This is shown in Figure 6. It is Black's turn so he is Player I. Black doubles. Should he? If he does, should White accept? If the cube is in the middle, we look in Table 8, row 6+1, column 1+1. Black should not double. If he does, White should accept. (This is correctly recommended by Tzannes and Tzannes.) Table 6 shows us that Black's expectation under best play, which means *not* doubling, is −17%. If instead Black has the cube, we use Tables 7 and 8. In this example we get exactly the same answer. This is not always the case, as we will see.

This example is also easy to analyze directly. If Black bears off in his next turn he will win. The chances are 15/36 (Table 1). If he does not bear off at once, White will win and Black will lose. So if the current stake is 1 unit, and Black does not double, Black's expected gain is +1 unit × 15/36 − 1 unit × 21/36 = −6/36 = $-16\frac{2}{3}\%$. Now suppose Black doubles and White accepts. Then Black's expected gain is +2 units × 15/36 − 2 units × 21/36 = −12/36 = −33%. On average, Black will lose an extra $16\frac{2}{3}\%$ of a unit if he makes the mistake of doubling and White accepts.

It's easy to see from this type of reasoning that if Player I has any two-man position and Player II will bear off on the next turn, then Player I should not double (if he can) when his chance to bear off in one roll is less than 50%. If his chance to bear off is more than 50%, he should double. Referring to the same Table 1 proves this rule which Tzannes and Tzannes cite for these special situations:

With double three, six–one, six–two
(or anything worse)

Keep dumb, hope for the best,
Anything better, don't delay,
Double the stakes with zest.

The Tzannes' and Tzannes Situation 73 is similar.

Here is a trickier situation that I do not think you could figure out without help from Tables 4–8. Suppose White has 6+6, Black has 4+4, White is on roll and the doubling cube is in the middle. This is shown in Figure 7. How does the game proceed for various rolls?

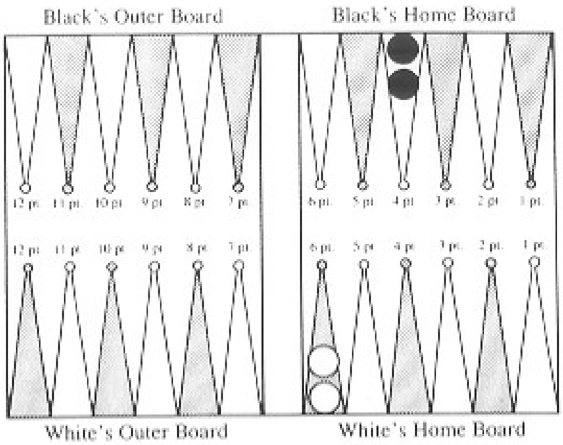


Figure 7. White to roll. The doubling cube is in the middle. Should White double? How does the game proceed for various rolls?

White is Player I. He consults Table 6 and sees his expectation is 16%. But Table 8 tells White not to double. We now show how to use the table to play optimally for a sample series of rolls. Suppose White rolls 3–1. How does he play it? He can end up with 6+2 or with 5+3.

The rule from Part I says that 5+3 looks better because it gives him a greater chance to bear off on the next turn. This is proven by the tables as follows: after White plays, it will be Black's turn. Black will be Player I with 4+4, White will be Player II with either 5+3 or 6+2. The cube will be in the middle. Which is best for White? Consult Table 3. We find Player I (Black) has an expectation of 88% if White has 5+3, whereas Black has 96% if White has 6+2. White wants to keep Black's expectation down so he plays to leave 5+3.

The situation after White makes this move is shown in Figure 8. Black is to roll and the cube is in the middle. Should Black double? Should White accept? Table 8 says Black should double and White should accept. Table 6 says Black's expected gain is 88% of the 1-unit stake. Next Black rolls 2–1. He can leave 4+1 or 3+2. The rule from Part I says 4+1 is better. To confirm this, note that after Black moves, White will be Player I with 5+3, Black will be Player II with either 4+1 or 3+2, and White will have the cube. Therefore

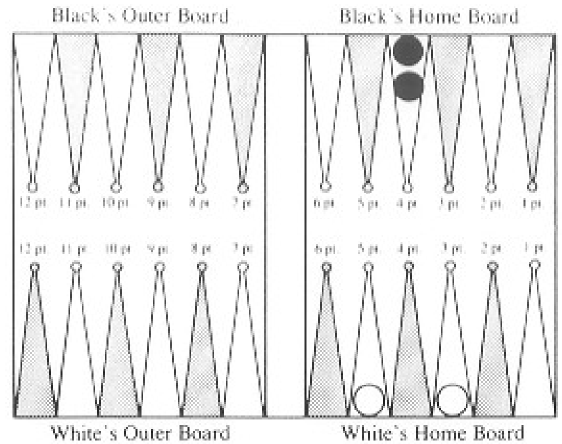


Figure 8. White did not double, then rolled 3–1 in Figure 2 and left 5–3. The cube is in the middle. It is now Black's turn. Black should double, White should accept, and Black's expectation is then 88% of the 1-unit stake.

we consult Table 7, not Table 6. If Black leaves 4+1, White's expectation is 2% of the current 2-unit stake. If Black leaves 3+2, White's expectation is 15%. Therefore Black leaves 4+1.

It is now White's turn. The situation is shown in Figure 9. The stake is 2 units, White's expectation is 2% of 2 units or 0.04 unit and White has the cube. What should he do? Table 8 tells us White should not double. White now rolls 5–2, leaving 1+0. Black does not have the cube. Table 5 gives his expectation as 61% of 2 units or 1.22 units. He wins or loses on this next roll.

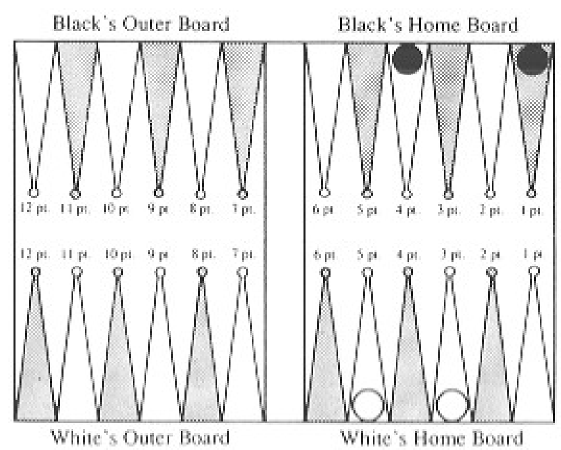


Figure 9. Black doubled, then rolled 2–1 in Figure 3 and left 4–1. The stake is 2 units, it is now White's turn, and he has the cube. White's expectation is 2% of the stake or 0.04 unit. What should White do?

The tables show certain patterns that help you to understand them better. For instance, for a given position it is best for Player I to have the cube. It is next best for Player I if the cube is in the middle and it is worst for Player I for Player II to have the cube. Therefore, for a given position, Player I's expectation is greatest in Table 7, least in Table 5, and in-between in Table 6. For instance, with Player I having 6+6 and Player II having 4+4, Player I's expectation is 25% if he has the cube, 16% if it is in the middle, and 7% if Player II has the cube. Sometimes two or even all three of the expectations are the same. For instance, if Player I has 6+6 and Player II has 6+5, Player I's expectation is 71% if he has the cube or if it is in the middle. If Player II has the cube, Player I's expectation drops to 36%.

Examination of the doubling strategies in Table 8 shows that the positions where Player I should double and Player II should fold are the same whether Player I has the cube or the cube is in the middle. Although this happens for the two-man end positions we are analyzing here, it is not always true in backgammon. The positions where Player I should double and it does not matter if Player II accepts or folds are also the same in Table 8. But some of the positions where Player I should double and Player II should accept are different. If Player I has the cube Table 8 shows that he should be more conservative. Intuitively, this is because if he has the cube and does not double, he prevents Player II from doubling, whereas if the cube is in the middle Player II cannot be prevented from doubling.

Table 8 leads to an example that will confound the intuition of almost all players. Suppose Player I has 5+2 and has the cube. Consider two cases (a) Player II has 1+0 and (b) Player II has 6+0. In which of these cases should Player I double? Clearly 6+0 is a worse position that 1+0. And the worse the position, the more likely we are to double, right? So of the four possible answers (double 1+0 and 6+0, double 1+0 but not 6+0, double 6+0 but not 1+0, do not double 1+0 or 6+0) we "know" we can eliminate "double 1+0, do not double 6+0", right? WRONG. The only correct answer, from Table 8, is: Double 1+0 but do not double 6+0. Try this on your expert friends. They will almost always be wrong. If they do get it right they probably were either "lucky" or read this column. In that case if you ask them to explain why their answer is correct, they probably will not be able to.

You may think that the loss would be slight by doubling 6+0 erroneously. But you have an expected gain of 29% by not doubling (Table 7), whereas by doubling it can be shown that your expectation drops to only 11%. The exact explanation is complex. The basic idea, though, is that if Player I doubles Player II, Player II accepts, and Player I does not win at once, Player II can use the cube against Player I with great effect at Player II's next turn.

After this was written, Don Smolen pointed out to me that Jacoby and Crawford (1973) discuss what is essentially the same example (they give Player II 4+1 instead of 6+0) on pages 116–117 of their excellent *The Backgammon Book*. Table 8 shows that essentially the same situation occurs

when Player I has 5+2 and Player II has 4+1, 5+0, 6+0, 2+2, or 3+2 and for no other two-man end positions.

Tables 5, 6, 7, and 8 present, for the first time anywhere, the complete exact solutions to two-man endgames in backgammon. The tables were calculated by a general method I have discovered for getting the complete exact solution to all backgammon positions that are pure races (i.e., the two sides are permanently out of contact). The intricate and difficult computer programs for computing Tables 5 through 8 were written by Don Smolen, so Tables 5 through 8 are our joint work. Don was a computer scientist at Temple University. He is now trading stock options on the floor of the American Stock Exchange. A skilled backgammon player, he won the 1977 American Stock Exchange Tournament.