DOES BASIC STRATEGY HAVE THE SAME EXPECTATION FOR EACH ROUND?

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Basic strategy in the game of blackjack is the strategy which is best on the first round of play, when there is only one player versus the dealer. Fundamental to the game are the questions: are basic strategy and its expectation to the player altered by (1) the presence of other players and the use of various strategies, or (2) by the round of play in which basic strategy is used. Under fairly general assumptions, we show that basic strategy and its expectation are invariant for the first m rounds of play. Here m is the largest number such that, in the specified setting, there will always be enough cards in the pack to complete the first m rounds of play.

For rounds $k > m$, we explain and illustrate why this changes. Next we look at the case of continuous play (if the pack runs out in mid-round, reshuffle the discards and continue dealing). The theory of finite Markov chains gives a qualitative description of what happens. We conjecture that in "realistic" examples, the chain has just one ergodic set and that it is regular. This is supported by examples and subsidiary theorems.

My thoughts were triggered by John Leib's article in *Blackjack Forum* and subsequent letters. An earlier version of this work appeared there as a response. This treatment owes much to Peter Griffin's seminal *Theory of Blackjack*.

1. INTRODUCTION: THE ISSUES

Does basic strategy have the same expectation on every round of play?

(D1) (Definition 1.) A strategy is *complete* if it provides a decision procedure for every situation.

We consider only complete strategies in this note.

(D2) A strategy is *myopic* if it only uses information about the cards as they are seen on the current round of play.

(D3) A myopic strategy is *simple* if it only uses information about the player's own cards and the dealers's cards, as they are seen, on the current round of play.

(D4) A strategy is *deterministic* if it always makes the same decision with the same cards and same information set.

We limit ourselves to deterministic strategies until stated otherwise. The usual card counting strategies are deterministic but not myopic.

Take a pack of n cards and label them $1, 2, \cdots, n$. These labels or "names" have nothing to do with the values of the cards for blackjack, or their possible indistinguishability under the rules, e.g. in a single deck ten value cards are typically indistinguishable under the rules of blackjack but all have distinct labels. The point of introducing the labels is that all cards are then distinguishable and each of the $n!$ possible orderings or shufflings is distinct from the others.

(A1) (Assumption 1.) Assume that the pack is randomly shuffled, i.e. that each of the possible $n!$ orderings (i.e. shufflings, or arrangements) of the n distinguishable labels has probability $1/n!$. Denote a typical ordering by $x_1, x_2, \cdots, x_n$ where x_1 is the label on the first card in the shuffled pack, x_2

the second, etc.

(D5). A strategy is *prescient* if it uses information about the order of the cards not yet played.

For example, an "anchor man" (Thorp, "Beat the Dealer") uses information received from the dealer to decide whether to draw or not. Another example is a dealer who peeks, then uses the information to alter the order of the cards by dealing seconds. All prescient strategies are ruled out in the note by (A1).

(D6). *Basic strategy* for a pack of cards, P, is the simple strategy with the maximum expectation on the first round of play, assuming only one player. If more than one simple strategy has this proper, then each is a *version* of basic strategy. Basic strategy obviously depends on P. Strictly speaking we should call it *P-basic strategy.* We will generally drop the P if it is clear from context.

We could have defined basic strategy as the best myopic strategy. But this can lead to impractically voluminous and complex strategies. Imagine seven players versus the dealer, with all cards visible except the dealer's hole card. Each decision in the correct optimal myopic strategy depends on which subset of cards has been seen, so far, on the current round. One can imagine an encyclopedic catalog, one set of strategy tables for each possible subset.

Typically, there is more than one version of basic strategy if and only if at least one strategy decision is an exact "push," i.e. there are two "best" alternatives with *exactly* the same expectation. With complex packs P, it seems unlikely this will happen. However it is easy to contrive simplistic examples where it does.

Example 1. (E1). Let $P = (10, 10, 10, 8, 3)$ in some order. Suppose the player has bean dealt (10,8) and the dealer shows 10. Does the player draw or stand? Either the dealer has a 10 under and a 3 is left or the dealer has a 3 under and a 10 is left. If the player hits, he wins with 21 versus 20 in one case and loses with 28 (busts) versus 13 in the other case. Since these are equally likely by (A1), his expectation is exactly 0. If instead the player stands, then he loses (18 versus 20) in one case and wins (18 versus 23, dealer busts) in the other. Again the expectation is exactly 0. Thus there are at least two versions of basic strategy for this P.

Question 1. (Q1) Is basic strategy still the optimal simple strategy on round 1 if there also are other players using non-prescient but otherwise arbitrary deterministic strategies? (We'll simply call these *arbitrary* from here on.) Whether basic strategy remains optimal or not, does its expectation remain the same?

Question 2. Is basic strategy on later rounds of play still the optimal simple strategy , assuming only one player on that round? Is the expectation the same?

Question 3. Is basic strategy on later rounds of play the optimal simple strategy when there are other players using arbitrary strategies on that round? Is the expectation the same?

2. FUNDAMENTAL INVARIANCE THEOREM FOR SIMPLE STRATEGIES

(A2) Assume that there are r_i players on round i, $i = 1, \cdots, m$. On each round the strategy of each player is specified and the order of the players is specified. The number of players and their strategies may vary from round to round. Strategies are arbitrary unless otherwise specified.

(A3m) Assume that m rounds can always be played.

(D7) The *strategy set* for a round of play is the ordered list of players and corresponding strategies, plus the dealer's strategy. The dealer's strategy is myopic so if each player has a myopic strategy then *the strategy set is myopic*.

A myopic strategy set has the important property that it always processes a given ordering in the same way no matter on what round of play the ordering appears, i.e. information gained on prior rounds does not affect play.

Consider a pack of n labeled cards with $n!$ orderings of the form $x_1, \cdots, x_n$. Suppose the strategy set for round 1 has only player 1 with simple strategy S. This (myopic) player plus dealer strategy set will use up a uniquely determined initial *segment* of the ordering, say $x_1, \cdots, x_t$. By *segment* we mean a consecutive sequence of cards in the ordering. Continuing over all $n!$ orderings we get a list of $n!$ of these initial segments. Each item in the list has probability $1/n!$ and typically the list has many copies of each such segment. Conversely, given the $n!$ segments, we can use the strategy set to calculate the outcome for each and the corresponding expectation for each. From this we get the expectation for the player's strategy for the given pack, on round 1.

In particular we have a list of segments and a probability measure defined thereon. This is a crucial notion for our theorem.

Theorem 1. (Fundamental invariance theorem for simple strategies.) Given (A1), (A2) and (A3m), any specified simple strategy S has a probability distribution on segments which is invariant, no matter on what round

k, $1 \leq k \leq m$, it is played, nor how many players there are on each round, nor what strategies the other players follow.

Proof. Consider two games played on the same pack of n cards. Game 1, or G_1, has m rounds with strategy sets $S_1, \cdots, S_m$. On round k, player i plays simple strategy S. A typical ordering is $x_1, \cdots, x_n$. In Game 2, or G_2, there is just one round with only one player. This player also uses strategy S. A typical ordering is $y_1, \cdots, y_n$. Let $f : x = x_1, \cdots, x_n \to y = y_1, \cdots, y_n$ be the mapping which selects the cards used by player i, round k of G_1, and those cards which the dealer needs to use in order to settle the hand of player i in G_1, and maps them into $A = y_1, \cdots, y_t$. If player i busts, the dealer in G_1 doesn't have to draw for player i, but may have to draw for other players who don't bust. These latter cards are not selected for A by f. Also if player i has a natural the dealer settles with player i without drawing any cards. Any cards he might draw because of other players are not selected for A. The order of cards in A is the same as the order in which they were used in G_1, so that they exactly suffice and the player has the same result in G_2 as in G_1. Preserve the order of the remaining cards in x and map them into $B = y_{t+1}, \cdots, y_n$.

This mapping is defined on every x because of (A3m) and the fact the strategy sets are deterministic. It is well-defined, i.e. each x maps onto one and only one y. Further, f is onto, i.e. every y corresponds to some x. To see this, pick any $y = (y_1, \cdots, y_n)$. Use game 2 to select an initial segment A. Call the rest B. Now use A,B to play game 1 as follows. On round k draw cards in order from A whenever the S player or the dealer need cards prior to settling the S player's hand. Also draw cards in order from B whenever other players or the dealer on round k, or anyone on other rounds, need cards.

Since f is onto and the number of x's and y's are the same, namely $n!$, then f is one-to-one whence the set of $f(x) = y$ are simply a shuffling or rearrangement of the x's. Therefore the list of sub orderings in G_1 which map into A's in G_2 is identical to the list of A's except for rearrangement. Therefore the probability distributions for S are identical in G_1 and G_2. Thus the probability distribution for S is always the same as that for round 1 with only one player, using S, versus the dealer. This proves the theorem.

Corollary 1.1. With the assumptions of Theorem 1, the probability distribution and expectation for any specified simple strategy on any round, whether or not there are other players with arbitrary strategies, is identical to that for one player versus the dealer "off the top of the pack."

Corollary 1.2. With the assumptions of Theorem 1, any specified version of "top of the deck basic strategy," with only one player versus the dealer, has invariant probability distribution and identical expectation no matter what round it is played, $1 \leq k \leq m$, no matter how many players there are on each round nor what strategies the players follow. Further, basic strategy is identical on all rounds.

Proof. From the definition of basic strategy and from Corollary 1, any specified version of "top of the deck" one player basic strategy has maximum expectation among simple strategies on any round so is a version of basic strategy on that round and conversely.

Basic strategy may be complicated because some decisions may depend on the values of the player's or dealer's cards seen so far during the round. See, e.g., Griffin (1999). Whether to hit hard 16 versus a 10 is often thus dependent. To simplify, such decisions may be fixed, e.g. choosing the total-dependent alternative that has the higher expectation, e.g. always hit hard 16 versus 10. This results in a loss in expectation that is generally slight. In this connection we note:

Corollary 1.3. With the assumptions of Theorem 1, the difference in expectation between basic strategy and a specified approximation to it is invariant.

Finally, under the assumptions of Theorem 1, the answers to Q1, Q2 and Q3 are all yes!

The observation in [Griffin, T.o.B., p.21] implied a different mapping than that used in Theorem 1, with somewhat different results. To see this, we first need to restrict (A2):

(A2a) Assume that there are r players on round i, $i = 1, \cdots, m$. On each round the strategy of each player and the order in which he plays is *the same*, and the strategy set is myopic.

Theorem 2. If (A1), (A2a), and (A3m) hold, then the probability distribution is identical for each of the first m rounds.

Proof. Consider the k th round, where $2 \leq k \leq m$. Suppose $x_1, \cdots, x_n$ is an ordering. Let A_1 be the cards used in round 1, A_2 the cards used in round 2, etc. Then we can abbreviate the ordering as $A_1, A_2, \cdots, A_k, C$ where C is everything left over after k rounds.

For each ordering $A_1, A_2, \cdots, A_{k-1}, A_k, C$ there is a different ordering $A_k, A_2, \cdots, A_{k-1}, A_1, C$ which we can pair with it. There are $n!/2$ such pairs. Restricting ourselves to one such pair, $A_1, A_2, \cdots, A_k, C$, and $A_k, A_2, \cdots,$ A_{k-1}, A_1, C we see that round 1 has probability $1/n!$ of using A_1 and $1/n!$ of using A_k. Now look at round k. The probability is $1/n!$ of using A_k and $1/n!$ of using A_1. Thus rounds 1 and k have identical probability distributions over each pair. All $n!$ orderings group into pairs by (A3m). Thus rounds 1 and k have identical probability distributions over the set of all orderings. Since $k = 2, 3, \cdots, m$, the probability distributions are the same (as for round 1) over the set of all orderings, for each of the rounds $1, 2, \cdots, m$.

Note that *any* rearrangement of an ordering is different because all cards are distinct, whether or not they have the same value for the game. Note too that (A2a) is required so that when we permute $A_1, \cdots, A_k$, they will still exactly suffice for a round of play.

Corollary 2.1. If (A1), (A2a) and (A3m) hold, and the strategy set consists of one player, who uses a simple strategy, the probability distribution and expectation is identical on the first m rounds.

Corollary 2.2. With the assumptions of Corollary 2.1, the optimal strategy on each of the first m rounds is the same, namely "top-of-the-deck" basic strategy, and the expectation is the same.

Note that Theorem 2 and its corollaries do not answer Q1 or Q3 because the results use (A2a), which fixes the number of players. The answer to Q2 is very restricted, also, because of (A2a).

Notice that the definitions, assumptions and proofs don't use the detailed rules of blackjack. Thus we have the generalization (some checking needs to be done to verify the mappings work and the definitions make sense):

Theorem 3. The preceding theorems and corollaries hold for other card games.

Example 2. When I studied it in the 60s, Nevada Baccarat was played typically from a pack of 8 decks or 416 cards. There was one "Player" hand and one "Banker" hand and the strategy for each was fixed so (A2a) held. A round used 4, 5 or 6 cards. The pack typically wasn't reshuffled until 400 or more cards had been used; 400/6=66.6 so with $m = 67$, assumption (A3m) was satisfied. All we need then is random shuffling, i.e. (A1), to conclude from Theorem 2 that the expectation on each of the first 67 rounds was identical for a bet on "Banker". The same was true for a bet on "Player".

Example 3. Woolworth Blackjack (Griffin, T.o.B. p.186). *Proof.* Verify (A1)-(A3m).

Example 4. "Wisconsin's" game (see Blackjack Forum, XII #4, p.51).

Example 5. Consider blackjack played from a pack of eight decks of 416 cards. Assume the player cannot resplit a pair, that the dealer is dealt both cards, and that he settles before the player draw if he is dealt a natural. Now suppose the player follows a strategy designed to use up as many cards as possible on each round. He can use up at most 31 + 31 pips per round: split a pair, draw to 21 on each, then hit each with a 10. In this case the dealer has at most 20 pips and the total number of pips used on the round is at most 82.

If the player chooses not to bust one of his two hands, he can use at most 21 + 31 pips. Then the dealer can use at most 26 pips: hit 16 with a 10. This variation only uses 78 pips. Thus at most 82 pips per round can be used.

There are 85 pips per suit per deck, if Aces equal one, which is their least efficient use, not 11. For 85*4 suits *8 decks we have a total of 2720 pips and 2720/82=33 with remainder 14. If we assume the deck will be shuffled whenever less than 82 pips remain, then we can choose $m = 33$. We conclude that top of the pack basic strategy is the best strategy for the first 33 rounds and that it has the identical expectation on each of the 33 rounds.

With only one player and no pair splitting, the player can draw a 10 to 21 and bust, using 31 pips. In this case the dealer can have at most 20, using at most 51 pips in the round. Alternatively, the player could draw up to a total of 21 and the dealer could hit 16 with a 10, and bust. This uses at most 47 pips. If the deck is reshuffled whenever less than 51 pips remain at the end of a round, then $m = 53$.

Note 1. In the preceding theorems, we established that the list of A's was the same for each round $k \leq m$ for, e.g., basic strategy. This means that the expectation gain for each basic strategy choice is exactly the same on every round.

3. WHEN SIMPLE STRATEGIES NEED NOT BE INVARIANT

Here we follow the lead of [Griffin, T.o.B. pp. 184-186].

(A4) Assume that a marker card is inserted between card t and card $t + 1$, where t is fixed. If a round uses card t, the pack is reshuffled at the end of that round. Further, t is chosen so that the last round will end on or before card n, which is the last card in the pack.

Note that this reshuffling rule is *deterministic*, i.e. for a given ordering and collection of strategy sets, the pack is always shuffled at the same point, and at the end of the same round. All play and all strategy decisions and all results will be identical. We'll extend our results later to include the more realistic case of a marker card whose location varies according to a specified probability distribution.

Let M be the maximum number of rounds that are possible and let m be the largest number such that m rounds can always be played. Then, given strategy sets $S_1, \cdots, S_M$ for rounds $1, \cdots, M$, any particular ordering may be written as follows: $x_1, \cdots, x_n = A_1, A_2, \cdots, A_k, B$ where A_i is the portion used on round i, and k is the last round played. B is the unused portion and $m \leq k \leq M$, with the value of k depending on the particular ordering $x_1, \cdots, x_n$.

(D8) A_i is the ith *segment* of the ordering.

Let T be the set of all $n!$ orderings. T is divided (partitioned) into the sets $T_m, T_{m+1}, \cdots, T_M$, where T_k is the set of all orderings having exactly k rounds of play before reshuffling. We could write $T = T_m + T_{m+1} + \cdots + T_M$.

Now suppose we have played m rounds and there are enough cards to continue. At this point we know our ordering was *not* in T_m. Instead it belongs to the smaller set $U_m = T_{m+1} + \cdots + T_M$. It seems plausible that the smaller set of orderings U_m generally does not have the same probability distribution as T, where for efficiency we lump together orderings which are equivalent under the rules of blackjack, when actually calculating the two probability distributions. If these distributions are different, it seems plausible that the basic strategy and corresponding expectation for round $m+1$ will generally be different. It also seems plausible that the original top of the deck basic strategy, and other simple strategies, will generally have different expectation.

Continuing, when we have completed round k, $m \leq k \leq M-1$, the ordering for the next, or $k+1st$, round belongs to the set $U_k = T_{k+1} + \cdots + T_M$. Note that the U_k get smaller and, we expect, probably more unlike T, as k

increases to $M-1$. (When $k=M$, we reshuffle so U_M has nothing in it.) In the simulation section below, we will indicate how to "statistically verify" some of these probabilities.

Intuitively, how might we expect top of the deck basic strategy to behave after we complete round m? Assume no pair splitting, at first, to simplify analysis. Consider T_m, the orderings that yield the fewest number of rounds. An average of somewhat more than t cards is used before reshuffling. So reaching the shuffle point t in relatively few rounds indicates relatively many cards were used per round. This suggests the cards dealt tended to be smaller than average. Therefore the cards remaining at reshuffle time, B, would tend to be larger than average. Blackjack theory suggests that the expectation on T_m is likely to be less than average, with a higher than average use of small cards, but the "count" is likely to be favorable, on average.

This picture shifts as k increases, until we reach the other extreme, T_M. When the maximum number of rounds M has been played, the number of cards per round is likely to be smaller than average. This suggests that the values of the cards used are larger than average and that the remaining cards are consequently smaller in value than average. This would tend to make the results per round more favorable than average but the "count" less favorable than average. The cards remaining in B at reshuffling time would tend to be smaller than average.

Aces are a special feature. They act "big" if they end up being valued as 11 and they act "small" if they end up being valued as 1. It may be that the effects roughly cancel, and that their effect in this analysis is minor. Pair splitting tends to use smaller cards, on average (it is rarely good to split tens). It also tends to increase the number of cards used on a round, hence decrease the number of rounds. That effect is greater with greater k because more pair candidates are likely to be dealt. This might tend to reduce differences in expectation between the T_k.

We make some of the foregoing a little more precise.

(A5) Assume that all strategy sets are myopic and identical and at least one player uses a simple strategy, P_s.

(D9) From the list of orderings for T_k, construct the list of all segments, one from each ordering, that are used on the ith round. Call this list $I_{k,i}$.

(D10) For a particular simple player strategy P_s in the strategy set S, let $E_{k,i}$ be the player's expectation on the list $I_{k,i}$. If p_k is the probability of the set T_k, namely the number of orderings in T_k divided by the total number of orderings $n!$ in T, then the expectation of P_s on any of the rounds $1 \leq i \leq m$ is $e(P_s, i) = p_m E_{m,i} + p_{m+1} E_{m+1,i} + \cdots + p_M E_{M,i}$. By Theorem 1, the $e(P_s, i)$ are all equal. Call this common value $e(P_s, 1)$.

Now look at rounds from m to M, $m \leq k \leq M$. We have $e(P_s, k) = (p_k E_{k,k} + \cdots + p_m E_{M,k})/(p_k + \cdots + p_M)$, valid for $m \leq k \leq M$. (Note $p_m + \cdots + p_M = 1$ so the two formulas agree if $i = k = m$.) Thus the formula for e varies and it seems plausible that the individual values of $E_m, \cdots, E_M$ will vary. Thus it is plausible that, generally, we expect the $e(P_s, k), k > m$, to be different from $e(P_s, 1)$.

Example 6. Consider "Woolworth" blackjack [Griffin, pp. 185-186], with a nine card deck having *two* fives (less calculation than three fives) and seven tens. There is no pair splitting or doubling (doubling never gains anyhow) and the player stands on 15 or more. A round of play will always use either four or five cards. Here are the possibilities: $(t,t;t,t)$, $(t,5;t,t)$, $(5,5,t;t,t)$, $(t,t;t,5,t)$, $(t,t;t,5,5)$, $(t,5;t,5,t)$, $(5,5,t;t,t)$. We have listed the player's cards followed by a semicolon, then the dealer's cards. The player's strategy does not depend on the dealer's up card in this simple example so it is equivalent to deal all the player's cards first, then the dealer's. If the first two cards for either dealer or player are $5,t$ or $t,5$ we count them as one possibility in the list, just to condense. Thus $m=1$ and $M=2$. Now we need an observation:

Equivalence classes and computational efficiency.

Under the rules of blackjack, 10,J,Q,K are all the same and suits don't matter. For an n card pack, the $n!$ orderings group into equivalent orderings that are identical for blackjack. For instance, in a standard 52 card pack, each equivalence class has $(4!)^9(16!)$ orderings. Thus there are 52! orderings but "only" $52!/((4!)^9(16!))$ equally probable equivalence classes. We could restate and derive all the preceding results using these equivalence classes, rather than the orderings. This reduces the computational complexity for applications. We'll assume this for the example, to which we now return. Corresponding to 9! orderings are $9!/((7!)(2!)) = 36$ equivalence classes.

Each equivalence class is described symbolically by an ordered 9-tuple of two 5s and seven ts (tens), e.g. $(t,t,5,t,5,t,t,t,t)$. There are, of course, 36 of these, with each equivalence class having probability 1/36. If we list the 36 ordered 9-tuples we find 20 have A_1 of length 4 so a second round is dealt, and so T_2 and U_1 each have 20 equivalence classes. The 16 remaining A_1 have length 5 so T_1 has 16 equivalence classes. The 20 members of T_2 are ten sequences beginning $(t,t,t,t,\cdots)$, five beginning $(t,5,t,t,\cdots)$, and five beginning $(5, t, t, t, \cdots)$. They are obviously distributed differently than the members of T_1. So are their A_1s.

The A_2s of T_2 are distributed differently than the A_1s of T, T_1, or T_2. There are 13 different A_1s; the three listed are in T_2, the other ten are only in T_1. The set of 13 A_1s happen to be the same as the set of 13 A_2s. However their probability distributions are not equal in any instance, as calculation verifies. A simple numerical argument also shows the two distributions are unequal, and gives another indication of why we might expect inequality in general: 8 of the A_1s have probability 1/36; the A_2s have probabilities of the form $0/20, 1/20, \cdots$. They are either too large or too small. When *can* $Prob(A_1) = Prob(A_2) > 0$? When $r/36 = s/20$ or $5r = 9s$, with r and s positive integers. But then 5 divides s and 9 divides r, so $r = 9r*, s = 5s*$ with r^* and $s*$ positive integers. But then $Prob(A_1) = 9r^*/36 = 5s^*/20 = Prob(A_2) = r*/4 = s^*/4$ so the common probability is 1/4, 2/4, 3/4 or 4/4. If there are more than four A_1s they can't all have such probabilities (sum of $Prob(A_1s) = 1$).

Calculation gives $E_{1,1} = 2/16,\ p_1 = 16/36 E_{2,1} = 0/20, p_2 = 20/36,$
$E_{2,2} = 2/20,\ e(P_s,1) = 2/36,\ e(P_s,2) = 2/20.$

Simulation 1. (Sim 1) We propose computer simulations to test some of the above. [See Griffin and Gwynn (1981)].

Choose a constant strategy set S of one player versus the dealer. Let the player strategy P be some simple strategy that approximates the basic strategy. Consider restricting pair splitting to either no resplitting or no splitting at all, to simplify. Choose t (the marker card) and the pack size so the $m \geq 2$ and $M \geq m + 2$.

For each k and i, record the number of events $N_{k,i}$ and the cumulative gain or loss, $A_{k,i}$. Then $\hat{E}_{k,i} = A_{k,i}/N_{k,i} \doteq E_{k,i}$.

(Q4) For each k value, how do the $\hat{E}_{k,i}$ vary?

(Q5) For each i, $1 \leq i \leq m$, are the estimates of the expectation of strategy P, $\hat{e}(P,i) = \hat{p}_m \hat{E}_{m,i} + \cdots + \hat{p}_M \hat{E}_{M,i}$ approximately the same, $1 \leq i \leq m$, as Theorem 1 predicts? (Test this result for statistical significance.)

(Q6) How do the estimates of the expectation of strategy P, $\hat{e}(P,k) = \hat{p}_k \hat{E}_{k,k} + \cdots + \hat{p}_M \hat{E}_{M,k}$ behave beyond m, i.e. $m \leq k \leq M$?

Warning: Simulations may not be as conclusive as many have believed. See [NYT] for a discussion of non-randomness in five popular computer programs for generating (pseudo) random numbers.

I believe that chaos theory gives us insight into how and why pseudo random number generators are non-random.

Variable marker card location. Suppose the t used to locate the marker card is random rather than fixed. Suppose the probability that $t = i$ is q_i, $a \leq i \leq b$, where $q_a + \cdots + q_b = 1$ and b is such that the last round always ends on or before card n, the last card in the pack. Then our results generalize; take results for $t = i$, multiply by q_i, and sum over i.

4. THE CASE OF CONTINUOUS PLAY

It is natural to look next at continuous play. By this we mean that the pack is shuffled and dealt as before and that we have assumptions (A1), (A3m) with $m = 1$, and (A5) without the requirement that one of the strategies be simple. But now instead of reshuffling when the remaining cards may not be sufficient to complete the round, we play "continuously" until the pack is exhausted. Then the discard pile is reshuffled and used to complete the round as needed. Then the next round is dealt, etc. (A3m) with $m = 1$ guarantees that the pack is large enough so it always is sufficient (under any ordering) to complete one round.

Griffin [T.o.B. pp.184-6, 200-1] shows that the theory of finite Markov chains is the appropriate description of what happens. In order to explore this further, we restate Griffin's ideas and introduce some definitions and notation.

Call any subset of the pack, distinct under the rules, a "state". In one deck blackjack, for instance, there are $5^9 \times 17 - 1 - 33,203,124$ such states. We subtract one because the full pack and the "empty" pack are equivalent for continuous play.

The first round of play begins with some initial state, s_1. Typically this is the full pack. This state has N_1 equally probable equivalence classes of orderings which are distinct under the rules. After the first round of play we are in some new state s_i which is uniquely determined by the particular ordering which occured. Let p_{1j} be the probability that if we start from state 1 we go to state j. Repeat for states $2, \cdots s$, where s is the total number of states. Then we have a finite Markov chain with a Markov transition matrix $M = (p_{ij})$, where $i, j = 1, \cdots, s$, $p_{ij} \geq 0$, and for each i $(\sum p_{ij} : j = 1, \cdots s) = 1$.

Let $x = (x_1, \cdots, x_s)$ be a row vector representing the probability distribution of the initial state. Specify one player, P. Let $e = (e_1, \cdots e_s)$ be the row vector such that e_i is the specified player's expectation on a round played from state i. Let e^t be the transpose (the same vector as a column). Then $xe^t = \sum x_i e_i$ is the player's expectation on round 1. At the start of round 2, the state distribution is xM so the expectation is xMe^t. Now the $n-1$ step transition matrix is M^{n-1} so the expectation on round n is $xM^{n-1}e^t$.

Each time the system returns to a state we need to have the same transition probabilities, i.e. constant p_{ij}. For this we need:

(A6) The strategy set is identical on every round.

Note that card-counting strategies are allowed because when we know the state we know the remaining cards so any such strategy will be the same. This yields our summary of Griffin:

Theorem 4. Under assumptions (A1), (A3m) with $m = 1$, and (A6), a continuous blackjack game has for a specified player the associated matrix M of state transition probabilities. M is the matrix of a finite Markov chain and the player's expectation on round n is given by $xM^{n-1}e^t$ where x is the initial state vector and e^t is the expectation vector for states. M and e depend on the strategy set.

It is well known [Kemeny, Snell pp. 35-38] that M has the following properties.

The states divide into equivalence classes where two states are equivalent "if they 'communicate', i.e. if it is possible to go from either state to the other one." Equivalence classes which can never be left are called *ergodic sets* and their states are called *ergodic states*. All other equivalence classes and their states are called *transient*. When the process leaves a transient set it never returns. The process ultimately enters an ergodic set and stays there.

There are two types of ergodic sets. Type I is called *regular*; if the process enters a regular state, then after a sufficiently long time, the process can be in any state of the regular ergodic set. This means that for some N and all

$k \geq 0$, m^{N+k} has all entries positive for the regular ergodic set. Further, the submatrix for the set tends to a matrix of identical rows. This means the corresponding expectation in continuous blackjack tends to a constant, given that the process has entered a regular ergodic set.

Type II ergodic sets are called *cyclic*. Such a set "has a period d, and its states are subdivided into d cyclic [sub] sets ($d > 1$). For a given starting position it will move through the cyclic [sub] sets in a definite order, returning to the [sub] set of the starting gate after d steps. ...after sufficient time... the process can be in any state of the cyclic [sub] set appropriate for the moment".

If there is a cyclic ergodic set and the process enters it, we would expect the expectation in Theorem 4 to, typically, oscillate, rather than approach a single limiting value. Of course, in a particular setting some or all of the multiple "limiting values" could all coincide, or be so near to each other that a simulation could not detect the difference.

If there is more than one ergodic set, say k of them, an initial state vector x has definite probabilities $q_1, q_2, \cdots, q_k$, where $q_1 + \cdots + q_k = 1$, of entering sets $1, \cdots, k$ respectively. Then the behavior is as described, with the probabilities q_i, depending on the nature of the sets.

Example 7. M is the 4×4 matrix with $M_{ij} = 1/2$ if ($i \leq 2$ and $j \geq 3$); or ($i \geq 3$ and $j \leq 2$); $M_{ij} = 0$ otherwise. Then M is cyclic of period 2; $M^{2n+1} = M$, $n = 1, 2, \cdots$; $M^{2n} = M^2$, $n = 1, 2, \cdots$.

(Q7) In blackjack, what kinds of sets can occur and in what combinations?

Assume (A1), (A3m), and (A6). Suppose the pack P has n cards. Assume

$m \le k \le M$ where k is the number of complete rounds that can be dealt, m is the minimum such k and M is the maximum. Let $x = A_1, A_2, \cdots, A_k, C$ be an ordering of the pack.

Let $X = a_1, a_2, \cdots, a_k, c$ be the corresponding equivalence class, where $a_1, \cdots, a_k, c$ are equivalence classes for $A_1, \cdots, A_k, C$. Here A_k is the last full round dealt. If A_k exhausts the pack, then C is the empty set. Otherwise C is not empty but is not sufficient to complete a round of play, hence the used cards are reshuffled in the midst of the round, with C in play.

Let X_1 be the equivalence class of the set of cards comprising $A_2, A_3, \cdots,$ A_k, C. Similarly for $X_2, \cdots, X_k$. If C is empty, let $X_k = P$, the equivalence class of the full pack.

Let S be the set of X_i that are generated from the strategy set by dealing each of the $n!$ orderings. Let E be the set of ergodic states, T the set of transient states, and $U = E + T$ the set of all states. S^c is the complement of the set S.

Lemma 1. S^c is a subset of T whence E is a subset of S.

Proof. After the pack is reshuffled, and the round is finished, that round is not in the pack, unless C was empty. Hence the state at the start of the next round is an X_1, or if C was empty, it is P. Hence we have entered S, never again to exit. Thus if we start in any state t not in S, we will have permanently entered S within $M + 1$ steps, after which t can never recur, so t must be transient.

We observe next that there are always plenty of transient states.

Lemma 2. Suppose there are r players. Then any state t derived from the full pack by deleting from 1 to $2r + 1$ cards is transient. Hence there are always transient states.

Proof. Note first that at least $2r + 2$ cards are used in a round of play. (This is true whether the dealer is dealt only one card before the players act or whether he is dealt two cards.) When we exhaust the pack, either we are in the midst of a round or at the end of a round. If we are at the end of a round then the next round is dealt from a full pack and it uses at least $2r+2$ cards so the resulting x_1 cannot be t. If we are in the midst of a round, then the next round is dealt from a pack with one round missing, so again the resulting state cannot be t.

Example 8. Blackjack, one player using basic strategy; no pair splitting; dealer stands on all 17s. Double down only on hard 9, 10, 11. If no aces appear on a round, then the minimum number of "pips" or "points" that can be used is $p(min) = 11 + 17 = 28$ and $p(max) = 21 + 26 = 47$. Thus any subset of P which contains all the aces, and has between 1 and 27 pips missing, must be a transient state. If no aces appear on the first two rounds, then by the end of round 2, at least 56 pips must be gone. Hence any subset of P which contains all the aces, and has between 48 and 55 pips missing, must be a transient state. Thus as many as 15 cards could be missing and the state could be transient.

After the pack is reshuffled, we know the state at the start of the next round is an X, or possibly P. But we don't know yet whether there are transient states in S or how many and what type of ergodic sets are in S.

Let $s*$ be the state corresponding to P.

Lemma 3. Given any s_o in S, if s_o is ergodic, all of its descendants are ergodic states in the same ergodic set.

Proof. A descendant of s_o is any state s that can be reached from it in a finite number of steps. An ergodic state can only move to a(n) (ergodic) state in the same ergodic set so s must be such. Note that is could conceivably take several shuffles to reach all the descendants of s_o.

Lemma 4. If $s*$ is ergodic then all the states of S are ergodic.

Proof. Since $m \leq k \leq M$, given any s in S, s can be reached from $s*$ within M steps, so the result follows from Lemma 3.

Lemma 5. If $s*$ is a descendant of each X_1, then $s*$ is ergodic, $E = S$, and E consists of a single ergodic set.

Proof. Every set in S is an X_1 or a descendant of an X_1 so we can go from $s*$ to any set in S. Conversely, starting from any s in S, we can go to an X_1 (wait for the reshuffle) and then, by the hypothesis, to $s*$. Therefore $s*$ and S are in a single Markov chain equivalence class of states. By Lemma 1, S contains ergodic states. Hence S is an ergodic set. Then by Lemma 1, $E = S$ and $s*$ is in E.

Example 9. [Griffin (1999) page 200 and Griffin and Gwynn (1981)]. The total number of states is 27. The set S has the eight states $A, \cdots, H$. $A, \cdots, F$ are of type X_1 and G, H are of type X_2. To verify that $S = E$ one can show that some power of the given 8×8 submatrix for S has only positive entries. Alternatively one can use Lemma 5. It is easily verified that $s*$ is a descendant of each of the six $X_1 s$.

Lemma 6. The ergodic states can be all regular or all cyclic; $s*$ can be either transient or ergodic.

Proof. We need examples of each instance. Choose Woolworth blackjack, with a pack of $n = 4$ tens. Number the four states $n, n-1, \cdots, 1$ according to how many cards remain. Then $s* = [4]$ is the (one-state) regular ergodic set and 3,2,1, are transient sets. For an example where the ergodic set is cyclic, choose $n = 8$. Then the ergodic set is [8,4] with cycle length two. The other six states are transient. For an example where the full pack is a transient state, choose $n = 5$. Then [1] is the only ergodic state; all other states including $s*$ are transient.

(Q8) Can there be two or more ergodic sets, or are there realistic examples with cyclic ergodic sets?

If the realistic examples typically have just one ergodic set and it is regular, then the preceding example of Griffin's is the typical state of affairs: as the round number j increases, the player's expectation E_j on round j may oscillate at first but will tend to a limiting value as j increases. Otherwise, with a cyclic ergodic set there can be multiple limiting values, and with more than one regular ergodic set there can be various limiting values with probabilities equal to those for entering each of the regular ergodic states.

Simulation 2. "Standard" blackjack. One player, basic strategy. Start the process with, say, a full deck, and deal continuously a very large number of hands. Repeat many times. Check the distribution of expectations to see if they're statistically consistent with a single limiting expectation. Test the overall expectation to see if it is statistically consistent with the top of the deck basic strategy expectation.

Simulation 3. Same as Simulation 2 except use small (random?) subsets of a full pack or packs. The object is to discover realistic examples of two or more ergodic sets or of more realistic cyclic ergodic sets, to answer Q8.

Conjecture. Realistic examples will typically have just one ergodic set and it is regular.

Lemma 5 strongly suggests this to us. We argue as follows. Suppose the pack is large enough so that several rounds (say 5 to 10) can always be dealt. Use Example 8 as a model. Then, start from any X_1 and construct a sequence $X_2, X_3, \cdots, X_k$, where k is the last complete round before reshuffling. With notation like that following Example 7, if C is empty we have returned to $s*$. If C is not empty, rearrange the cards in $x = A_1, \cdots, A_k, C$ so that either more cards are used on rounds $1, \cdots, k$ and C becomes empty, or so that fewer cards are used and C becomes exactly sufficient for round $k+1$ and $x = B_1, B_2, \cdots, B_{k+1}, D$ with D empty. In either case we have returned to $s*$. It seems intuitively plausible that if k is large enough, we will be able to do this for every X_1. If that is true, then Lemma 5 establishes the conjecture.

Note that we can start the process in any state by using a random device. For instance, with a pack of n cards we can display and "burn" the first k cards, $k = 0, 1, \cdots, n-1$, with probabilities $1/n$.

We have made some progress in settling the conjecture. For example, we have proven:

Theorem. Suppose the pack consists of n decks, for any $n = 1, 2, \cdots$, with the Aces removed. Suppose the dealer stands on all total of 17 or more. Suppose the player cannot split pairs and can double down only on totals of 9 or higher. Suppose there is only one player and that player always uses basic strategy (for the given pack). Assume (A1).

Then the conjecture is true: there is just one ergodic set and it is regular.

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