The Mathematics of Gambling

by Edward O. Thorp

"Did you see Ken Uston on TV? He's amazing. He can count every card in the deck," said my friend David Menkes. I had missed the program, so Menkes described Uston's demonstration. Menkes said that Uston asked the host, David Hartman, to remove one card from the deck. Then, after thumbing through the remaining cards, he identified the missing card.

"Do you think I could ever learn to do that?" asked Menkes. I thought for a few seconds and said, "I can teach you that trick in three minutes." (Plan on 15 minutes, including practice.) Menkes said "Show me." I did and I'll show you now.

The Missing Card

Take a full deck of cards. Shuffle it and remove one card face down. The objective is to find the blackjack value of the missing card. Take the remainder of the pack and add the cards one at a time according to their blackjack value. For instance, if the first few cards were A, K, 7, 3, 9, 5, your count goes A=1, +K=11, +7=18, +3=21, +9=30, +5=35, and so on. Note that A=1 and 10, J, Q, K all count as 10. The total of all cards in the single deck is 340. Suppose your total is 333. Then the missing cards must be $340 - 333=7$.

Too hard? Of course. Here's an easier way. You only need to know the last digit of the total to find the missing card. If the last digit is a 3, the missing card is 7. If it is zero, the missing card is a 10 (value), and so forth. Thus your simplified count for A, K, 7, 3, 9, 5 above goes A=1, +K=1, +7=8, +3=1, +9=0, +5=5, and so on. You "cast out 10s." Note that 10, J, Q, K can be counted as zero, which speeds things up even more.

8s and 9s are the hardest to add on to a previous total. You can use a trick to speed the count up more: As $9=10 - 1=0 - 1=-1$, on positive counts you subtract 1 instead of adding 9. Similarly, $8=10 - 2=0 - 2=-2$; instead of adding 8, you can subtract 2. Notice that this method of keeping only the last digit of the total is the same one used to add baccarat hands. You could call this system of adding "baccarat addition." Mathematicians call it "addition modulo 10."

"How did you know that trick?" asked Menkes. "Who do you think Ken learned it from?" I responded. Uston and I had met in June 1975 at Lake Tahoe, where I spoke at the Second Annual Conference on Gambling. During the extensive socializing that went on, I showed Uston and many others the above card counting trick. It was great fun to first amaze them with my "incredible card counting skill" and then to explain the trick.

Determining Suit and Rank

If you found the blackjack trick easy, here's a somewhat harder one. Suppose someone says to you, "I want to know more than the blackjack value of the card. If it's a ten-value card, I also want to know whether it's a 10, J, Q, or K." Proceed as before—shuffle the deck and remove one card. Now count the remaining cards as follows: A=1, 2=2, . . . 9=9, 10=10, J=11, Q=12, K=13. When you total cards, cast out 13s instead of 10s. (That's what makes this trick more difficult.) Counted this way, one deck totals $364=28 \times 13$. This means that if we count an entire deck and cast out all the 13s, we get a total of zero. If a card is missing, we examine the total we get. Say it is 8. Then 5 is required to get 13=0, so the missing card must be a 5. Noting that $K=13=0$, $Q=12=13 - 1=-1$, and $J=11=13 - 2=-2$ will make counting easier.

Here is a much harder trick, especially valuable for bridge players. Take one card out of the deck, count through the remaining cards, and identify the missing cards by rank and by suit. Here's how. First suppose you only wanted to know the suit. The bridge suits in order of increasing rank are C (clubs), D (diamonds), H (hearts), and S (spades). Count $C=1$, $D=2$, $H=3$, $S=4$ and cast out 4s. Thus $S=4=0$, and spades can be ignored. Then the suit count for a full deck is $1 \times 13 + 2 \times 13 + 3 \times 13=78=4 \times 19 +2$, so casting out nineteen 4s leaves a total of 2 for an entire pack. Now remove one card and count the remainder, casting out 4s. If you find your total is 2, then the missing card must count 0 to make the full deck total 2. Therefore it is a spade. A total of 1 means a card of suit value 1—a club—is missing. A total of 0 indicates the missing card has suit value 2, a diamond. A total of 3 tells you the missing card is a 3 (so the full deck will total $3 + 3 =6=6 - 4 =2$), which is a heart.

To determine the rank of the missing card, simply count through the deck another time, using the "casting out 13s" technique explained previously. However, if you only want to count through the deck once, which is harder but more impressive, proceed as follows.

Keep track of two totals as you use both casting out techniques to count the cards. List the totals as a pair of numbers (rank total, suit total)=(X,Y). Suppose the cards begin AD (ace of diamonds), 3C, KD, 8S, 4H, and 9S. The counting goes like this: start at (0,0); see AD and add (1,2), a 1 to the first number and a 2 to the second, to get (1,2); see 3C and add 3 to the first number and 1 to the second to get (4,3); see KD and add 13=0 in the first number and 2 to the second to get (4,5)=(4,1), because we cast out 4s from the second number; see 8S and add (8,0) to get (12,1); see 4H and add (4,3) to get (16,4)=(3,0); see 9S and add (9,0) to get (12,0) and so forth. Suppose your final total is (7,3). Then the rank is 6 and the suit is 3. The missing card is the six of hearts.

It should be evident that these tricks are not directly useful in playing blackjack, though they do provide a good test of whether you'll find card counting in blackjack easy to learn. However, there are useful methods, and I will discuss some of them in future columns.

Counting With a Computer

If you have a hard time counting past ten with your shoes on, don't despair—help is on the way. I've designed a program for the HP-65 hand-held computer that will count cards, compute ratios, and adjust the size of bets using Braun's HI-OPT I system.

That system assigns the following point values to cards as they are used: 3, 4, 5, and 6 (low cards) are +1; 2, 7, 8, 9 (intermediate cards) equal 0; and tens (10, J, Q, K) are −1. In this version, aces are counted separately. For strategy decisions, the "true count" is $C = 52 \times (\textit{sum of points})/N$, where N is the total number of cards left. The true count C is very similar to the high-low index used in the complete point count of *Beat the Dealer*, revised edition. In fact, if you take the complete point count strategy tables and divide all index values by 2, the results are a satisfactory approximation to the HI-OPT I strategy tables.

Programming the HI-OPT I

Keys	Comment
0	
STO 1	L=0
STO 2	Z=0
STO 3	−H=0
4	use 8 for two decks, etc.
STO 4	A=4
5	use 104 for
2	two decks, etc.
STO 5	N=52

Keys	Comment
R/S	

Keys	Comment
LBL	Key A
A	Reduces
RCL 5	N
1	when
−	pressed to
STO 5	N − 1
RCL 1	then L
1	goes
+	to
STO 1	L + 1
LBL	then
1	finds C:
RCL 1	L
RCL 3	−H
+	P points
RCL 5	N
5	
2	
÷	N/52
÷	52 P/N=C
RTN	

Keys	Comment
LBL	Key
B	B
RCL 5	reduces N
1	
−	to
STO 5	N − 1
RCL 2	then Z
1	goes
+	to
STO 2	Z + 1
GTO	then
1	finds C

Keys	Comment
LBL	key
C	C
RCL 5	reduces N
1	to
−	
STO 5	N − 1
RCL 2	then Z
1	goes
+	to
STO 2	Z + 1
GTO	then
1	finds C

Keys	Comment
LBL	key
D	D
RCL 5	reduces N
1	to
−	
STO 5	N − 1
RCL 4	then A
1	goes
−	to
STO 4	A − 1
GTO	then
1	finds C

Keys	Comment
LBL	key
E	E
RCL 4	A
RCL 5	N
÷	finds A/N
1	
ENTER	
1	
3	
÷	finds 1/13
−	
64	
×	64(A/N − 1/13)
RCL 1	L
RCL 3	−H
+	P=L − H
RCL 5	N
5	
2	
÷	N/52
÷	C
+	B
RTN	ready

A good count for strategy decisions may not be as good for bet size determinations and vice versa, as Peter Griffin explains in his new book, *The Theory of Blackjack*. Because aces have a relatively small effect on many strategy decisions, it's useful to count them as zero for strategy, but keep a separate side count for betting decisions. The program I set up does this. The formula I use for correcting the count is $B = C + (A/N - 1/13) \times 64$, where A is the number of aces left and B is the number of units to bet.

To bet as accurately as possible, first divide your capital into 200 units. Bet one unit if $B \leq 1$ and bet the nearest whole number of units if $B > 1$. Of course, it may not be wise to consistently and mechanically vary your bets this way. But when $B > 1.5$, try to bet at least two units in order to get more money down in those favorable situations.

To use: enter card, press RTN, R/S. As cards appear, press key A if you see low cards 3, 4, 5, 6. Total low cards seen and points L are stored in register R_1. Press key B when you see middle cards 2, 7, 8, 9; the number seen is Z, stored in register R_2. Press key C for high cards (ten-values). The number indicated is stored in register R_3 with a minus sign, so R_3 gives high card points as $-H$. Press key D for aces that you see; these are subtracted from R_4. Because R_4 is set initially at four aces, it always contains the remaining number A of unseen aces. The current number of unseen cards N is stored in R_5 and is revised automatically whenever keys A, B, C or D are pressed.

After each card is counted, the display shows the true count C, used for strategy decisions. You can also use C for betting decisions, but it's much better to use B, the true count adjusted to consider the ace count.

For example, suppose the cards dealt on the first hand are, in order, ace (player's first card), 7 (dealer's up card), 6 (player's second card), 5 (player's first hit), 3 (player's second hit), 5 (player's third hit for a total of 20), 9 (dealer's down card), 3 (dealer hits for a total of 19). Then we count ace, key D, display $C=0.00$; 7, key B, display $C=0.00$; 6, key A, display $C=1.06$; 5, key A, display $C=2.17$; 3, key A, display $C=3.32$; 5, key A, display $C=4.52$; 9, key B, display $C=4.62$; 3, key A, display $C=5.91$.

To determine the bet size for the second hand, you could use $C=5.91$ and bet six units. Or you could hit key E, display 5.35, and bet five units. To see the status of the deck, we can examine registers R_1, R_2, R_3, R_4 and R_5 respectively by hitting RCL 1, RCL 2, and so on. At this point in the example, we find $L=5$, $Z=2$, $H=0$, $A=3$, and $N=44$.

The only changes required for two or more decks are shown in the program at the fifth, seventh and eighth instructions. If you are comfortable with using a small programmable calculator, you should be able to easily modify this program to count cards for most of the other point count systems, including multiparameter approaches like David Heath's. 9

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ing and playing over the shoulder of seated players. All tables, that is, except a dozen $25 minimum tables, where the dealers stood with arms folded and the decks spread out before them. I told the pitboss that in Vegas, the minimum bets at the tables were fluctuated to accommodate the mood of the crowds, and that if he'd drop the price to $5, these tables also would fill up very fast. The pitboss explained that empty $25 tables at noon would fill up and show a larger profit by closing time than the full $5 tables. He was right—the $25 tables were packed all evening.

By 1 p.m., I got a seat and told my wife I'd quit at 6 p.m. for dinner. I wouldn't say I had a cold seat, but I won only about 40 percent of the time, and it took me four hours to get a blackjack. The player next to me got a blackjack about every ten minutes and the player next to him about every twenty minutes. The fourth player continually drew a pair of aces, split, and got 3s and 4s to complete and lose both hands. Seldom was I able to hit a stiff hand without busting, although the dealer did so regularly. I warned people behind me not to play over my shoulder, as I was not winning. I should have heeded my own advice, gotten up and left. But I kept hoping for an improvement. The player next to me lived near the casino and dropped in daily to grind out $100 and leave. However, my luck didn't improve, and by 6 p.m., I was $200 down. I wished I had promised to meet my wife earlier. After dinner and a walk along the Boardwalk, I was ready for one more fling. I found all tables full; even the $100 minimum tables were three-deep in players. I walked from table to table, discouraged and dejected, for about an hour. Finally, a first-base spot opened up at a $25 table, and I took it and won a couple of hands. I tried to "add and drag" as I won, but my win streaks seemed limited to one or two in a row. Finally, I had two green chips down ($50) and a double down situation occurred, so I bet $100. I won, picked up my $200, got up and left. That kind of play was just too rich for my blood. According to an eyewitness, it was too rich for another player who reportedly lost $60,000 in 30 minutes.

Casinos love impulse bettors, which is why I resisted the one impulse bet I was sorely tempted to make. Again while I was walking and watching, the third-base slot opened up at a $25 table. "Play one hand for $500," an inner voice screamed at me. In over 30 hours' play in Vegas and Atlantic City, this was the only impulse I experienced. "Never bet more than you can afford to lose" and "Never bet so much on a hand that that hand becomes crucial" I remembered from books I had read on the subject. While I was fighting this impulse, another player came up, and I graciously stepped aside and let him have the seat. You guessed it, the dealer dealt him a blackjack. I walked out, not playing another hand. Obviously, my timing was bad, I was trying too hard, it was late and I was tired. I quit with a meager $135 total win in Atlantic City. With my $300 from Las Vegas, this wasn't enough to pay for my vacation, but it did cover about a third of it.

The next morning, after deciding not to try one more time, we headed for home. We were surprised to find Philadelphia was less than an hour from Atlantic City and Chicago was less than a two-day trip by auto. In other words, Atlantic City casinos are readily accessible to the eastern half of the United States. That ease of access could lead to a reduction in business for Las Vegas casinos.

Which place do I like best, Atlantic City or Las Vegas? Frankly, my roots are in Vegas. I've played thousands of hours over hundreds of trips there and I've only played about 15 hours in Atlantic City. For the next several years, I will prefer the Vegas junkets, but after Resorts has expanded and Caesars World, Playboy, and others have established their casinos in Atlantic City, I may have another fling there. Meanwhile, it's "Las Vegas, here I come, right back where I started from." 9