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CONCAVE UTILITIES ARE DISTINGUISHED BY THEIR OPTIMAL STRATEGIES

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1. INTRODUCTION

Mossin [5], Thorp [7], and Samuelson [6] showed for specific pairs of utility functions that different utilities can lead to different optimal strategies. In particular the optimal investment strategy for the utility $\log x$ is not necessarily the optimal strategy for the utility $\frac{1}{\gamma} x^\gamma$ $(\gamma \neq 0)$.

These examples suggest the following generalization, of obvious importance to general utility theory.

Consider a T stage investment process. At each stage allocate resources among the available investments. Each chosen sequence A of allocations (''strategy'') yields a corresponding terminal probability distribution F_T^A of assets at the completion of stage T. For each utility function $U(\cdot)$, consider those strategies $A^*(U)$ which maximize the expected value $\int U(x) dF_T^A(x)$ of terminal utility. Assume sufficient hypotheses on U and the set of F_T^A so that the integral is defined and that furthermore the maximizing strategy $A^*(U)$ exists. Then is it true in general that $A^*(U_1)$ is not $A^*(U_2)$ for "distinct" utilities U_1 and U_2?

As we now show, the answer is yes: the Mossin – Thorp – Samuelson results for specific utility pairs generalizes to the principal class of interest in modern utility theory.

2. THE MAIN THEOREM

We prove this for the class of "interesting" concave utilities. We begin with more special hypotheses.

Theorem 1. *Let U and V be utilities defined and differentiable on $(0, \infty)$ with $U'(x)$ and $V'(x)$ positive and strictly decreasing as x increases. Then if U and V are inequivalent, there is a one period investment setting such that U and V have distinct sets of optimal strategies. Furthermore, the investment setting may be chosen to consist only of cash and a two-valued random investment, in which case the optimal strategies are unique.*

Corollary 2. *If the utilities U and V have the same (sets of) optimal strategies for each finite sequence of investment settings, then U and V are equivalent.*

Two utilities U_1 and U_2 are equivalent if and only if there are constants a and b such that $U_2(x) = aU_1(x) + b$ $(a > 0)$, otherwise U_1 and U_2 are inequivalent.

Let X_i $(1 \leqslant i \leqslant k)$ be the (random) outcome per unit invested in the ith "security". We call $(X_1, \ldots, X_k)$ the investment setting. We assume X_i is independent of the amount invested. Let the initial capital be Z_0 and let the final capital be Z_1. A strategy is an allocation $W = (w_1, \ldots, w_k)$ where w_i is the fraction of Z_0 allocated to security i. We assume $w_i \geqslant 0$ for all i, that $\sum_i w_i = 1$, and that wealth is infinitely divisible. Thus the w_i may assume any real values consistent with the constraints and with the requirement that $\sum_i w_i X_i$ is in the domain of the utility function U.

Given a particular U satisfying the hypotheses of the theorem, suppose $\mathsf{E}U(Z_1(W))$ is maximized by some strategy W^*. Then W^* is an *optimal* (or *best*) *strategy* for U relative to the given investment setting.

Proof of Theorem. Suppose that U and V have the same optimal strategies for every one period investment setting consisting of cash and a two-valued random investment. It will be shown that U and V are equivalent, which will establish the logical contrapositive to the theorem and hence the theorem itself.

In the proof of theorems we shall assume for technical simplicity that the initial capital $Z_0 = 1$. When theorems have been established for this case, consideration of the transformation $U_0(s) = U(Z_0 s) = U(t)$ gives the theorems for arbitrary $Z_0 > 0$. We shall therefore state the general results without further comment after proving the $Z_0 = 1$ case.

Let the only investment (besides cash) be X where $\mathsf{P}(X = 1 - b) = = q = 1 - p$ and $\mathsf{P}(X = 1 + a) = p$, where $a > 0$ and $0 < p, b < 1$. The choice $0 < b < 1$, rather than simply $b = 1$, has been made because for $b = 1$ and $w = 1$, the expression $U(0)$ would arise and 0 is not necessarily in the domain of U (e.g., $U(x) = \log x$). The available strategies are to allocate the fraction w of recources to X and $1 - w$ to cash, with $0 \leqslant w \leqslant 1$.

At the end of the period, we have

$$\mathsf{E}U(Z_1(w)) = pU(1 + aw) + qU(1 - bw) = f(w)\,. \tag{2.1}$$

To find the maximum, consider $f'(w) = apU'(1 + aw) - bqU'(1 - bw)$. Since $U'(t)$ strictly decreases as t increases, we have $f'(w)$ decreasing strictly as w increases. Thus there is a unique maximum. If $f'(w^*) = 0$ for some w^* with $0 \leqslant w^* \leqslant 1$, then the maximum is at this unique w^*. If $f'(w) > 0$ for all w with $0 \leqslant w \leqslant 1$, then the unique maximum is at $w = 1$. If instead $f'(w) < 0$ for $0 \leqslant w \leqslant 1$, then the unique maximum is at $w = 0$.

If $f'(w) = 0$ we have $\dfrac{U'(1+aw)}{U'(1-bw)} = \dfrac{bq}{ap}$. Suppose $a > 0$ and $\frac{1}{2} < b < 1$ are given and we wish $f'\left(\frac{1}{2b}\right) = 0$. Letting $\lambda = \dfrac{U'\left(1+\frac{a}{2b}\right)}{U'\left(\frac{1}{2}\right)}$, we can solve $\lambda = \dfrac{bq}{ap}$ for p, with $0 < p < 1$. Thus for each $a > 0$ there is a choice of p, hence an X, such that $w^* = \dfrac{1}{2b}$ is optimal for U.

Now suppose that U and V have the same optimal strategies for all such investment settings. Then $w^* = \dfrac{1}{2b}$ for V also and we have $\dfrac{U'\left(1+\frac{a}{2b}\right)}{U'\left(\frac{1}{2}\right)} = \dfrac{V'\left(1+\frac{a}{2b}\right)}{V'\left(\frac{1}{2}\right)}$ for all $a > 0$. Letting $V'\left(\frac{1}{2}\right) = \alpha U'\left(\frac{1}{2}\right)$ we find $V'(t) = \alpha U'(t)$ $(t > 1)$ whence $V(t) = \alpha U(t) + \beta$ $(t > 1)$.

When $t < 1$, we proceed similarly. Choose X so that $\mathsf{P}(X = 2) = p$ and $\mathsf{P}(X = 1 - b) = q$, where $0 < b < 1$. Then

$$\mathsf{E}U(Z_1(w)) = pU(1+w) + qU(1-bw) = f(w)$$

$$f'(w) = pU'(1+w) - bqU'(1-bw)$$

and the maximum is unique and located as before.

If $f'(w) = 0$ we have $\lambda = \dfrac{U'(1-aw)}{U'(1+w)} = \dfrac{p}{aq}$ and given $w = b$, $0 < b < 1$, we can choose p with $0 < p < 1$ such that $\lambda = \dfrac{p}{aq}$. Then as before we find $V'(1-ab) = \gamma U'(1-ab)$ and since a and b can be any numbers such that $0 < a$, $b < 1$, then $V'(t) = \gamma U'(t)$ $(0 < t < 1)$ where $\gamma = \dfrac{V'(1+b)}{U'(1+b)}$. But γ was shown to be α.

Thus $V(t) = \alpha U(t) + \delta$ $(0 < t < 1)$. Also $V(1) = \alpha U(1) + \epsilon$. Hence $V(t) - \alpha U(t) = \beta$ if $t > 1$, δ if $t < 1$ and ϵ if $t = 1$. But $V(t) - \alpha U(t)$ is continuous so $\beta = \delta = \epsilon$ so $V(t) = \alpha U(t) + \beta$. Thus U and

V are equivalent under the assumption that they have the same optimal strategies for all one period investment settings containing only (cash and) a two-valued random investment. The logical contrapositive assertion is the Theorem. This completes the proof. The Corollary follows a fortiori.

Note that a single investment setting of the type in the proof will not in general distinguish inequivalent utility functions. For instance, if $\mathsf{E}(X) \leqslant \leqslant 0$ then $w = 0$ is the unique optimal strategy for all the utilities of Theorem 1 (more generally, for all strictly concave utilities, as defined below) so such X distinguish between none of these utilities. It may be of interest to characterize each investment setting by the pairs of utility functions it distinguishes between or "separates", and to similarly characterize collections of investment settings.

For a security X, let $m(X)$ and $M(X)$ be the greatest and least numbers, respectively, such that $\mathsf{P}(m(X) \leqslant X \leqslant M(X)) = 1$. Then for a collection C of investment settings whose securities are $\{X_\alpha : \alpha \in A\}$, where A is some index set, let $m_A = \inf\{m(X_\alpha) : \alpha \in A\}$ and $M_A = \sup\{M(X_\alpha) : \alpha \in A\}$. Evidently, if $U(t) = V(t)$ for $m_A \leqslant t \leqslant M_A$, the collection C will not separate U and V. Thus a collection with $m_A = 0$ and $M_A = \infty$ will be needed in general to prove the conclusion of Theorem 1.

Next we generalize Theorem 1 to concave non-decreasing utilities defined on $(0, \infty)$. We do not make the common assumption that first or even second derivatives exist. A function f is *concave* on an interval I if for each pair of points $x_1 \neq x_2$ in I and each number s with $0 < s < 1$, then $f(sx_1 + (1-s)x_2) \geqslant sf(x_1) + (1-s)f(x_2)$. If $f(sx_1 + (1-s)x_2) > sf(x_1) + (1-s)f(x_2)$ always, then f is *strictly concave.* (We use "concave" to mean "concave from below".)

The more general definition includes such computationally and empirically natural functions as the "polygonal" utilities. In these, the utility is a sequence of linear segments. The vertices are such that the function lies on or below each segment extended, and the ordinates of the vertices increase as the abscissas increase.

First, recall some facts from the elementary theory of concave functions. (Most texts give results for convex functions. But f is concave exactly when $-f$ is convex so the theories of concave and convex functions are equivalent.) A concave function is either continuous in the interior of its domain or non-measurable. An increasing function is always measurable so our utilities are continuous. A continuous concave function f defined on an open interval has a left derivative f'_- and a right derivative f'_+ defined everywhere. (If the left endpoint a is included in the interval of definition, then $f'_-(a)$ is not defined and $f'_+(a)$ may or may not be defined. Similarly, if the right endpoint b is included in the interval of definition, then $f'_+(b)$ is not defined and $f'_-(b)$ may or may not be defined.) Furthermore, $f'_-(t) \geqslant f'_+(t)$ for all t except the endpoints in the domain of f and whenever $t_1 < t_2$ then $f'_-(t_1) \geqslant f'_-(t_2)$ and $f'_+(t_1) \geqslant f'_+(t_2)$. There are at most countably many points where $f'_-(t) >$ $> f'_+(t)$; otherwise $f'_-(t) = f'_+(t) = f'(t)$ and f is differentiable. Proofs of these assertions and further theorems on concave functions are given for instance in Hardy, Littlewood, Polya [3].

Theorem 3. *Let U and V be concave utilities defined on $(0, \infty)$, one of which is strictly increasing on $(0, 1 + e)$ for some $e > 0$. If U and V are inequivalent then there is a one period investment setting such that the sets of optimal strategies for U and for V are distinct. The investment setting may be chosen to consist only of cash and a two-valued random investment. If U and V are each strictly concave on the same one of the sets $(0, Z_0]$ or $[Z_0, \infty)$, then the optimal strategies are unique and U and V therefore have distinct optimal strategies.*

Proof. We proceed as in the proof of Theorem 1 until we obtain equiation (2.1).

Note that f is concave and that if U is strictly concave on either $(0, 1]$ or $[1, \infty)$ then f is strictly concave. Now $f(w)$ is a continuous function defined on the closed bounded set $\{w\colon 0 \leqslant w \leqslant 1\}$ hence f has an absolute maximum. Let w^* be a point where f attains its maximum. It follows from the continuity of f that the set of all such w^* is closed.

From the concavity of f, the set of points w^* where f attains its maximum is also convex, hence it is a closed interval in $[0, 1]$. If f is strictly concave the maximum is unique.

For any w^* with $0 < w^* < 1$, f is a maximum if and only if $f'_-(w^*) \geqslant 0 \geqslant f'_+(w^*)$. A maximum occurs at $w^* = 0$ if and only if $f'_+(0) \leqslant 0$. A maximum occurs at $w^* = 1$ if and only if $f'_-(1) \geqslant 0$. If the maxima occur on an interval $[a, b]$ with $0 \leqslant a < b \leqslant 1$, then $f'_-(a) \geqslant 0$ and $f'_+(a) = 0$, $f'_-(b) = 0$ and $f'_+(b) \leqslant 0$, and $f'(w^*)$ exists and is zero for $a < w^* < b$.

Equation (2.1) yields

$$\begin{aligned} f'_-(w) &= apU'_-(1 + aw) - bqU'_-(1 - bw) \geqslant \\ &\geqslant apU'_+(1 + aw) - bqU'_+(1 - bw) = f'_+(w)\,. \end{aligned} \tag{2.2}$$

Since $U'_-(t)$ and $U'_+(t)$ are non-increasing as t increases, it follows from equation (2.2) that $f'_-(w)$ and $f'_+(w)$ are non-increasing as w increases.

Let c be such that $0 < c < b$ and $U'(1 - c)$ and $V'(1 - c)$ are defined. This is possible because U' and V' are both defined except at countably many points hence there are uncountably many points in $(0, 1)$ where both U' and V' exist. With a and b already given, choose $w = \frac{c}{b}$. Consider now the case where $U'_-\left(1 + \frac{ac}{b}\right) > 0$. Then we may choose p with $0 < p < 1$ in equation (2.2) so that $f'_-\left(\frac{c}{b}\right) = 0$. This means $f'_+\left(\frac{c}{b}\right) \leqslant 0$ and since $w = \frac{c}{b}$ is not an endpoint of $[0, 1]$ this means f attains its maximum at $\frac{c}{b}$, thus $\frac{c}{b}$ is optimal for U in the given investment setting.

Since U and V have the same optimal strategies, $w = \frac{c}{b}$ is optimal for V hence V attains its maximum there so for $w = \frac{c}{b}$, $g'_-(w) = apV'_-(1 + aw) - bqV'_-(1 - bw) \geqslant 0$ and $apV'_+(1 + aw) -$

$-bqV'_+(1-bw) = g'_+(w) \leqslant 0$. Note that $g'_-\left(\frac{c}{b}\right) \geqslant 0$ and the fact $V'_-(1-c) > 0$ implies that $V'_-\left(1+\frac{ac}{b}\right) > 0$. We may show similarly that if $V'_-\left(1+\frac{ac}{b}\right) > 0$ then $U'_-\left(1+\frac{ac}{b}\right) > 0$. Since a is chosen independently of b and c this means that for each $t > 1$, $U'_-(t) > 0$ if and only if $V'_-(t) > 0$. But this is readily shown to be equivalent to the statement that $\{t\colon U(t) = \sup U(t)\} = \{t\colon V(t) = \sup V(t)\}$, i.e. that if either U or V become horizontal for $t \geqslant e > 1$ then they both become horizontal for $t \geqslant e > 1$. For $t > e$, we have of course $U'(t) = V'(t) = 0$. For $t < e$, the argument continues as follows.

From $f'_-(w) = 0$, $apU'_-\left(1+\frac{ac}{b}\right) = bqU'(1-c)$, noting that $U'_-(1-c) = U'(1-c)$. Thus $\dfrac{U'_-\left(1+\frac{ac}{b}\right)}{U'(1-c)} = \dfrac{bq}{ap}$. From $g'_-(w) \geqslant 0$, it follows similarly that $\dfrac{V'_-\left(1+\frac{ac}{b}\right)}{V'(1-c)} \geqslant \dfrac{bq}{ap}$. Letting $\alpha = \dfrac{V'(1-c)}{U'(1-c)}$ yields

$V'_-\left(1+\frac{ac}{b}\right) \geqslant \alpha U'_-\left(1+\frac{ac}{b}\right)$. Since the choices of b and c were independent of that a, the result holds for all $a > 0$, therefore $V'_-(t) \geqslant \alpha U'_-(t)$ for all $t > 1$.

A similar argument shows that $V'_+(t) \leqslant \alpha U'_+(t)$ for all $t > 1$. Thus, except for at most countably many points, $V'(t) = \alpha U'(t)$ for $t > 1$. Now U and V are readily shown to be absolutely continuous on any closed subinterval of $(1, \infty)$, as a consequence of the fact they are continuous, concave, and non-decreasing, thus $V - \alpha U$ is absolutely continuous. The absolute continuity of $U - \alpha V$ and the fact that $(V - \alpha U)' = 0$ almost everywhere implies that $V - \alpha U = \beta$, a constant (Goffman [2], p. 242, Prop. 12).

A similar argument shows that $V(t) = \alpha U(t) + \gamma$ for $t < 1$. The role of 2 in the proof of Theorem 1 is played by any number c such that $1 < c < e$ and $U'(c)$ and $V'(c)$ are both defined. One then shows as in the proof of Theorem 1 that $V(t) = \alpha U(t) + \beta$ for $0 < t < \infty$. We have established the contrapositive assertion as in the proof of Theorem 1. This completes the proof.

The hypothesis that either U or V (hence both, from the proof) is strictly increasing for a positive distance to the right of 1 is required. If instead U and V are merely concave and non-decreasing, the conclusion of Theorem 3 need not hold. For instance, let $U(t) = V(t) = 0$ if $t \geqslant d$, where $0 < d \leqslant 1$. Let $U(t)$ and $V(t)$ each be extended to $(0, d)$ so that they are continuous, concave, and strictly increasing on $(0, d)$. Then all such utilities have the same optimal strategies, yet many pairs are inequivalent.

To obtain an inequivalent pair, let $U(t) = t - d$ if $0 < t < d$ and let $V(t) = -(t - d)^2$. If for some constants α and β, $V(t) = \alpha U(t) + \beta$ then $V'(t) = \alpha U'(t)$. But $V'(t) = -2(t - d) \not\equiv \alpha = \alpha U'(t)$.

To see that all such utilities U have the same optimal strategies, note that $W = (w_1, \dots, w_k)$ is optimal for the investment setting $(X_1, \dots \dots, X_k)$ if and only if $\mathrm{P}\left(\sum_i w_i X_i \geqslant d\right) = 1$, in which case $\mathrm{E}U(Z_1(W)) = = 0$. If instead $\mathrm{P}\left(\sum_i w_i X_i < d\right) > 0$ then for some $\epsilon > 0$, $\mathrm{P}\left(\sum_i w_i X_i \leqslant \leqslant d - \epsilon\right) = \delta > 0$. Then $\mathrm{E}U(Z_1(W)) \leqslant \delta U(d - \epsilon) < 0$ so W is not optimal.

3. OTHER SEPARATING FAMILIES

We next establish the conclusion of Theorem 1 using investment settings with n points in their range. We determine the effect of varying the payoffs $(x_1, \dots, x_n)$ and their probabilities $(p_1, \dots, p_n)$ separately. One surprising conclusion (part (b)) can be stated in terms of an example. Suppose X consists of betting on a wheel of fortune divided into red, white and blue sectors, with payoffs of $\frac{1}{2}, \frac{3}{2}$, and $\frac{3}{4}$ respectively. Then if U and V are inequivalent on $\left[\frac{1}{2}, \frac{3}{2}\right]$ the areas of the sectors may be chosen so U and V have distinct optimal strategies. But if the wheel is divided into just red and blue sectors, with payoffs of $\frac{1}{2}$ and $\frac{3}{2}$, then there are two inequivalent utilities on $\left[\frac{1}{2}, \frac{3}{2}\right]$ which have the same optimal strategies for every choice of areas for the two sectors.

Theorem 4. *Suppose U and V are increasing strictly concave utilities on $(0, \infty)$. Let X be a random variable with outcomes $0 \leqslant x_1 \leqslant \leqslant x_2 \leqslant \ldots \leqslant x_n$ with $x_1 < 1$ and $x_n > 1$. Suppose $\mathsf{P}(X = x_i) = p_i > 0$, $\sum_{i=1}^{n} p_i = 1$.*

(a) *Let n and the p_i be given. If U and V have the same optimal strategies for each X (i.e. $x_1, \ldots, x_n$ vary), then U and V are equivalent.*

(b) *Let n and the x_i be given. Suppose U' and V' exist and are continuous at 1. If U and V have the same optimal strategies for each X (i.e. $p_1, \ldots, p_n$ vary) and at least three x_i are unequal to 1, then U and V are equivalent on $[Z_0 x_1, Z_0 x_n]$. If exactly two of the x_i's are unequal to one, there are utilities U and V which are not equivalent on $[Z_0 x_1, Z_0 x_n]$, but which have the same optimal strategy for each X.*

Proof. Assume $Z_0 = 1$. Let $R = X - 1$ and $r_i = x_i - 1$. Then investing w in X gives an expected return (with respect to U) of $\mathsf{E}(U(wR + 1)) = \sum_{i=1}^{n} p_i U(wr_i + 1)$. Each function $U(wr_i + 1)$ is differentiable except at a countable set C_i of points, so except for w in the countable set $C_1 \cup \ldots \cup C_n$ the expectation $\mathsf{E}(U(wR + 1))$ is differentiable at w with $\frac{d\mathsf{E}(U(wR + 1)}{dw} = \sum_{i=1}^{n} p_i r_i U'(wr_i + 1)$. Similarly each function $V(wr_i + 1)$ is differentiable except at a countable set. Thus, except at a countable set D of points in $[0, \infty]$ both $\mathsf{E}(U(wR + 1))$ and $\mathsf{E}(V(wR + 1))$ are differentiable functions of w. They are also strictly concave functions of w.

For part (a) let $p_1, \ldots, p_n$ be given and choose w_0 in $(0, 1) - D$. Consider the vectors $\alpha = (U'(w_0 r_1 + 1), \ldots, U'(w_0 r_n + 1))$ and $\beta = = (V'(w_0 r_1 + 1), \ldots, V'(w_0 r_n + 1))$. Suppose that the non-zero vector

$\gamma = (c_1, c_2, \ldots, c_n)$ is perpendicular to α, i.e., the inner product $(\alpha, \gamma) = 0$. Choose $r_i = \dfrac{c_i}{p_i\left\{\epsilon \sum_{j=1}^{n} |c_j|\right\}}$ with $\epsilon = \max \dfrac{1}{p_i}$, $1 \leqslant i \leqslant n$.

Since each component of α is positive, some $c_i > 0$ and some $c_j < 0$, hence some $x_i > 1$ and some $x_j < 1$. Also $r_i + 1 = x_i > 0$. Then $\dfrac{d\mathsf{E}(U(wR + 1))}{dw}\Big|_{w=w_0} = \sum_{i=1}^{n} p_i r_i U'(w_0 r_i + 1) = 0$ and $\mathsf{E}(U(wR + 1))$

has a maximum at w_0. By hypothesis $\mathsf{E}(V(wR + 1))$ has a maximum at w_0 and, since it is differentiable there, $\dfrac{d\mathsf{E}(V(wR + 1))}{dw}\Big|_{w=w_0} =$

$= \sum_{i=1}^{n} p_i r_i V'(w_0 r_i + 1) = 0$ i.e., $(\beta, \gamma) = 0$. Hence the set of vectors perpendicular to α is also perpendicular to β which implies that $\beta = a\alpha$. Since the components of α and β are non-negative, $a \geqslant 0$. Equating components

$$(3.1) \qquad U'(w_0 r_i + 1) = aV'(w_0 r_i + 1)$$

where a is a non-negative function of $r_1, \ldots, r_n$ and w_0. Since U and V are strictly concave there is a point t_0 not in D, $w_0 < t_0 < 1$, with $V'(t_0) > 0$ and $U'(t_0) > 0$. Choose r_1 so that $w_0 r_1 + 1 = t_0$, choose $r_2 > 0$ with $t = w_0 r_2 + 1$ not in D, and choose $r_3 < \ldots < r_n$ so they are not in D. Then $U'(t_0) = a(r_1, \ldots, r_n, w_0)V'(t_0)$ and $U'(w_2 r_2 + 1) = a(r_1, \ldots, r_n, w_0)V'(w_0 r_2 + 1)$. Thus $a = \dfrac{U'(t_0)}{V'(t_0)} > 0$ is constant. So $V'(t) = aU'(t)$ for any $t > 1$ not in D. Since V and U are absolutely continuous on any closed subinterval, $V(t) = aU(t) + b$ for all $t > 1$. A similar argument shows that $V(t) = cU(t) + d$ for $t < 1$ with $c = \dfrac{V'(t_0)}{U'(t_0)} = a$. The equivalence of U and V now follows (as in the proof of Theorem 1) from their continuity.

For part (b) suppose that the x_i are given, with $0 < x_1 < x_2 < \ldots \ldots < x_p \leqslant 1 \leqslant x_{p+1} < \ldots < x_n$. We proceed as before, but now consider, for $0 < w_0 < 1$ and U, V differentiable at $w_0 r_j + 1$, $1 \leqslant j \leqslant n$, the vectors $\tilde{\alpha} = (r_1 U'(w_0 r_1 + 1), \ldots, r_n U'(w_0 r_n + 1))$ and $\tilde{\beta} = (r_1 V'(w_0 r_1 + 1), \ldots, r_n V'(w_0 r_n + 1))$. Since $\tilde{\alpha}$ has both positive and negative components there is a vector $(d_1, d_2, \ldots, d_n)$ perpendicular to $\tilde{\alpha}$ with each $d_i > 1$. Choose $p_i = \dfrac{d_i}{\sum_{i=1}^{n} d_j}$, thus $p_i > 0$ and $\sum_i p_i = 1$, and define X by $\mathrm{P}(X = x_i) = p_i$. Thus

$$0 = (\tilde{\alpha}, (d_1, d_2, \ldots, d_n)) = \sum_{i=1}^{n} \frac{d_i}{\sum_{i=1}^{n} d_j} r_i U'(w_0 r_i + 1) =$$

$$= \frac{d\mathrm{E}(U(wR + 1))}{dw}\Big|_{w = w_0}$$

By hypothesis $\dfrac{d\mathrm{E}(U(wR + 1))}{dw}\Big|_{w = w_0} = \dfrac{(\tilde{\beta}, (d_1, \ldots, d_n))}{\sum_i d_j} = 0$ so $(\tilde{\beta}, (d_1, \ldots, d_n)) = 0$. Suppose that $\tilde{\gamma} = (e_1, e_2, \ldots, e_n)$ is perpendicular to $\tilde{\alpha}$. Let $d_0 > \max|e_i|$ and choose $p_i = \dfrac{e_i + d_0 d_i}{\sum_{i=1}^{n} e_j + d_0 d_j}$. Note that

$p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$. Thus

$$\frac{d\mathrm{E}(U(wR + 1))}{dw}\Big|_{w = w_0} = \sum_{i=1}^{n} p_i [r_i U'(w_0 r_i + 1)]$$

and letting $D = \sum_{j=1}^{n} (e_j + d_0 d_j)$ gives

$$\frac{1}{D} \sum_{j=1}^{n} e_i r_i U'(w_0 r_i + 1) + \frac{1}{D} d_0 \sum_i d_i r_i U'(w_0 r_i + 1) =$$

$$= \frac{1}{D} (\tilde{\alpha}, \tilde{\gamma}) + \frac{1}{D} d_0 (\tilde{\alpha}, (d_1, \ldots, d_n)) = 0\,.$$

Hence, $\left.\frac{d\mathrm{E}(V(wR+1))}{dw}\right|_{w=w_0} = 0$. This yields $(\tilde{\beta}|\tilde{\gamma}) + d_0(\tilde{\beta},(d_1,\ldots,d_n)) = (\tilde{\beta},\tilde{\gamma}) = 0$. Thus

$$U'(w_0 r_i + 1) = a(p_1, p_2, \ldots, p_n, w_0)V'(w_0 r_i + 1) \quad (1 \leqslant i \leqslant n). \tag{3.2}$$

For w in $(0, 1)$ with U, V differentiable at $wr_i + 1$ $(1 \leqslant i \leqslant n)$ we have (3.2) with $w_0 = w$. Consider the quotient

$$h(w) = a(p_1, \ldots, p_n, w) = \frac{U'(wr_i + 1)}{V'(wr_i + 1)} \qquad (1 \leqslant i \leqslant n). \tag{3.3}$$

First look at the case where at least three x_i's are unequal to one. Suppose that $x_1 < 1 < x_{n-1} < x_n$; the proof where two or more points fall to the left of 1 is similar.

Let $\varphi_i(w) = wr_i + 1$. The countable collection of functions $\{U, V, U \circ \varphi_n^{-1}, V \circ \varphi_n^{-1}, U \circ \varphi_n^{-1} \circ \varphi_{n-1} \circ \varphi_n^{-1}, V \circ \varphi_n^{-1} \circ \varphi_{n-1} \circ \varphi_n^{-1}, U \circ \varphi_n^{-1} \circ \varphi_{n-1} \circ \varphi_n^{-1} \circ \varphi_{n-1} \circ \varphi_n^{-1}, V \circ \varphi_n^{-1} \circ \varphi_{n-1} \circ \varphi_n^{-1} \circ \varphi_{n-1} \circ \circ \varphi_n^{-1}, \ldots\}$ is simultaneously differentiable except at a countable set of points D_0 in $(0, 1)$.

Choose t in $(1, x_n) - D_0$ and write $t = w_1 r_n + 1$, so $w_1 = \varphi_n^{-1}(t)$, and set $t_1 = w_1 r_{n-1} + 1 = \varphi_{n-1}[\varphi_n^{-1}(t)]$. We can also write $t_1 = w_2 r_n + 1$; $w_2 = \varphi_n^{-1}(t_1)$. By (3.3) we have $h(w_1) = h(w_2)$, since U and V are differentiable at w_1 and w_2. Note that $w_2 < w_1$, in fact, $w_2 = \lambda w_1$ with $\lambda = \frac{r_{n-1}}{r_n}$. Setting $t_2 = w_2 r_{n-1} + 1 = \varphi_{n-1}(w_2)$, $t_2 = w_3 r_n + 1$ and $w_3 = \varphi_n^{-1}(t_2)$. Then $h(w_2) = h(w_3)$ since U and V are differentiable at $w_2 = \varphi_n^{-1} \circ \varphi_{n-1} \circ \varphi_n^{-1}(t)$ and at $w_3 = \varphi_n^{-1} \circ \circ \varphi_{n-1} \circ \varphi_n^{-1} \circ \varphi_{n-1} \circ \varphi_n^{-1}(t)$. Continuing inductively $t_j = w_j r_{n-1} + 1 = w_{j+1} r_n + 1$ and $w_{j+1} = \lambda w_j$. Iterating this equation $w_{j+1} = \lambda^j w_1 \to 0$ as $j \to \infty$, thus $h(w_1) = \ldots = h(w_n) \to h(1)$ since U' and V' are continuous at 1. Hence the equation $\frac{U'(t)}{V'(t)} = h(1)$ holds except for countably many t in $(1, x_n)$ and thus, since U and V are absolutely continuous on any closed subinterval, $U(t) = h(1)V(t) + c$ for all t in $[1, x_n)$.

Let t belong to $(x_1, 1)$ with U and V differentiable at $wr_j + 1$, $1 \leqslant j \leqslant n$. Then $t = wr_1 + 1$ and from equation (3.3) $\frac{U'(t)}{V'(t)} = \frac{U'(wr_n + 1)}{V'(wr_n + 1)} = h(1)$. Since U and V are absolutely continuous on closed subintervals of $(x_1, 1)$, $U(t) = h(1)V(t) + d$. The continuity of U and V at 1 implies that $c = d$ and thus U and V are equivalent on $[x_1, x_n]$.

To complete the proof we must consider the case where there are only two x_i's distinct from one, say, $0 < x_1 < 1 < x_2$. Let g_0 be any non-constant positive function on $[1, x_2]$ with a continuous derivative which is zero at 1. Define g on $[x_1, 1]$ by $g(wr_1 + 1) = g(wr_2 + 1)$ for $0 \leqslant w \leqslant 1$. Choose a so that $\max\limits_{x_1 \leqslant t \leqslant x_2} |g'(t)| - a \cdot \min\limits_{x_1 \leqslant t \leqslant x_2} |g(t)| <$ < 0 and define $U(t) = \int_0^t e^{-at} dt = \frac{1 - e^{-at}}{a}$ and $V(t) = \int_0^t e^{-at} g(t) dt$. Because $U''(t) = -ae^{-at} < 0$ and $V''(t) = e^{-at}(g'(t) - ag(t)) < 0$, U and V are strictly concave. Also $U'(t) = e^{-at}$ and $V'(t) = e^{-at} g(t)$ are positive so U and V are strictly increasing. Clearly U and V are not equivalent on $[x_1, x_2]$.

For these two functions U and V and $0 < w < 1$,

$$\frac{d\mathrm{E}(V(wR + 1))}{dw} = r_1 p_1 V'(wr_1 + 1) + r_2 p_2 V'(wr_2 + 1) =$$

$$= r_1 p_1 g(wr_1 + 1) U'(wr_1 + 1) + r_2 p_2 g(wr_2 + 1) U'(wr_2 + 1) =$$

$$= g(wr_1 + 1) \frac{d\mathrm{E}(U(wR + 1))}{dw}.$$

Hence $\frac{d\mathrm{E}(U(wR + 1))}{dw} = 0$ if and only if $\frac{d\mathrm{E}(V(wR + 1))}{dw} = 0$, and so w_0, $(0 < w_0 < 1)$, is an optimal strategy for U (with respect to X) if and only if it is an optimal strategy for V. If the derivative $\frac{d\mathrm{E}(U(wR + 1))}{dw}$ is never 0, the equation above shows that it has the same sign as $\frac{d\mathrm{E}(V(wR + 1))}{dw}$; so 0 (or 1) is an optimal strategy for U if and only if it is an optimal strategy for V.

We have seen that U and V are two utilities on $[x_1, x_2]$ which are not equivalent, but which have the same optimal strategies for all random variables with outcomes x_1 and x_2.

Remark. Our proofs may be modified readily to prove the theorems when U and V are defined on the *closed* interval $[0, \infty)$ and also when the interval is (c, ∞) or $[c, \infty)$, with $c < Z_0$. Presumably $c > 0$. (Alternately, the $[c, \infty)$ result implies the (c, ∞) result: if $U(x) = V(x)$ on every interval $[c + \epsilon, \infty)$ $(0 < \epsilon < Z_0 - c)$ then $U(x) = V(x)$ on (c, ∞).)

4. QUESTIONS FOR FURTHER INVESTIGATION

Friedman – Savage [1] and Markowitz [4] have shown that utilities which are not everywhere concave are of interest. This leads us to a question which we have not been able to answer yet:

Is the class of utilities which are continuous and strictly increasing (and differentiable everywhere, bounded, and even strictly positive derivative, if you like) distinguished by their optimal strategies?

In the real world factors such as human error, the discreteness of assets and monetary units, etc. make it in general not possible to choose the optimal allocation $W^* = (w_1^*, \ldots, w_k^*)$. The continuity of the utility in conjunction with boundedness of the *attainable* utilities implies that "sufficiently small" deviations from W^* will ensure that the realized utility is "close" to the optimum.

One feels as well that in the real world, the exact values of the utility function should not be critical. In other words, if two utility functions are somehow "close," the consequences of choosing one rather than the other should be "close."

What should it mean for two utility functions to be "close?" First, observe that we must define closeness not for functions, but for equivalence classes of functions. Let U be a utility. The equivalence class of U, written $[U]$, is the set $\{V\colon V = \alpha U + \beta,\ \alpha > 0\}$. For the class β of bounded utilities, i.e., $M(U) \equiv \sup U(t) < \infty$, $m(U) \equiv \inf U(t) > -\infty$, we suggest that each $[U]$ equivalence class be represented by $\widetilde{U} = \dfrac{U-M}{M-m} + 1$. Note that $M(\widetilde{U}) = 1$ and $m(\widetilde{U}) = 0$. Then the "closeness" of U and V, i.e., of $[U]$ and $[V]$, is defined to be $\sup[\widetilde{U}(t) - \widetilde{V}(t)]$ and written either $d(U, V)$ or $d([U],[V])$ or $d(\widetilde{U}, \widetilde{V})$.

We now show that U and V can be "close" yet the optimal strategies for U and V need not be. For $n \geqslant 2$, let $\widetilde{U}_n$ and $\widetilde{V}_n$ be defined as follows:

$$\widetilde{U}_n(t) = \frac{2nt}{n+1} - 1 \quad \text{if} \quad 0 \leqslant t \leqslant 1 + \frac{1}{n} \quad \text{and} \quad 1 \quad \text{if} \quad t > 1 + \frac{1}{n};$$

$$\widetilde{V}_n(t) = \frac{2n-1}{n+1}\, t - 1 \quad \text{if} \quad 0 \leqslant t \leqslant 1 + \frac{1}{n},$$

$$\frac{t+n-3}{n-1} \quad \text{if} \quad 1 + \frac{1}{n} < t \leqslant 2, \quad \text{and} \quad 1 \quad \text{if} \quad t > 2.$$

Then $d(\widetilde{U}_n, \widetilde{V}_n) = \frac{1}{n}$. Now choose an investment setting consisting only of cash and the security X, where $\mathrm{P}(X = 1 - \epsilon) = q$, $\mathrm{P}(X = 1 + a) = p$, $\frac{1}{n} < a < 1$, and $0 < \epsilon, p, q < 1$. Assume $Z_0 = 1$. A calculation shows that if $ap > q\epsilon \dfrac{(2n-1)(n-1)}{n+1}$, then the unique optimal strategy for U_n is $w^* = \dfrac{1}{an}$ and for V_n the unique optimal strategy is $w^* = 1$.

Thus for any $\delta > 0$ we can construct sequences $\widetilde{U}_n$ and $\widetilde{V}_n$ such that $d(\widetilde{U}_n, \widetilde{V}_n) \to 0$ as $n \to \infty$ and $|w^*(\widetilde{V}_n) - w^*(\widetilde{U}_n)| \geqslant 1 - \delta$, where $w^*(\widetilde{U})$ means an optimal strategy for $\widetilde{U}$.

Even though a small "error" in the utility function can lead to a large change in optimal strategy, it can only lead to a small change in consequences, in the following sense. (We use the abbreviation $U(W)$ for $\mathrm{E}U\big(Z_0 \sum_i w_i X_i\big)$. Thus for each W, $U(W)$ is a number and $U\big(Z_0 \sum_i w_i X_i\big)$ is a random variable.)

Lemma. *If* $d(\widetilde{U}, \widetilde{V})$ *is "small," then* $\widetilde{U}(W^*(\widetilde{V})) \doteq \widetilde{U}(W^*(\widetilde{U}))$ *and* $\widetilde{V}(W^*(\widetilde{U})) \doteq \widetilde{V}(W^*(\widetilde{V}))$, *i.e., if* U *and* V *are "close," an optimal strategy for one is "nearly optimal" for the other.*

Proof. Let $d(\widetilde{U}, \widetilde{V}) \leqslant \epsilon$ so $\widetilde{V}(t) + \epsilon \geqslant \widetilde{U}(t)$. Then for any allocation W, $\widetilde{V}\big(Z_0 \sum_i w_i X_i\big) + \epsilon \geqslant \widetilde{U}\big(Z_0 \sum_i w_i X_i\big)$ and $\mathrm{E}\big(\widetilde{V}\big(Z_0 \sum_i w_i X_i\big) + \epsilon\big) = \mathrm{E}\big(\widetilde{V}\big(Z_0 \sum_i w_i X_i\big)\big) + \epsilon \geqslant \mathrm{E}\widetilde{U}\big(Z_0 \sum_i w_i X_i\big)$, or $\widetilde{V}(W) + \epsilon \geqslant \widetilde{U}(W)$. Interchanging $\widetilde{U}$ and $\widetilde{V}$ in the argument yields $\widetilde{U}(W) + \epsilon \geqslant \widetilde{V}(W)$ so $|\widetilde{U}(W) - \widetilde{V}(W)| \leqslant \epsilon$. The choices for W of $W^*(\widetilde{U})$ and $W^*(\widetilde{V})$ yield the conclusion of the lemma.

The lemma and the example show us what may happen if we replace a U by a nearby V which may have more desirable properties, such as differentiability (of various orders), strictly increasing, etc.: The optimal strategies may change drastically but the maximum utility over all strategies changes only slightly.

Note added in proof: The authors have since extended the central result of the paper, Theorem 3, as follows.

Theorem. *Let* U *and* V *be continuous non-decreasing functions defined on an arbitrary interval* I *of the real line. Then if* U *and* V *are inequivalent, there is a one-period two security investment setting such that* U *and* V *have distinct optimal strategies if either* (a) U *and* V *are in the class of all functions which are either concave or convex, or* (b) U *and* V *are in the class of all functions with a second derivative which exists and is continuous, except perhaps for a set of isolated points.*

Thus the Theorem includes the utility functions generally encountered.

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