COMMON STOCK VOLATILITIES IN OPTION FORMULAS

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UNIVERSITY OF CALIFORNIA
IRVINE

PROCEEDINGS

SEMINAR ON THE ANALYSIS OF SECURITY PRICES

MAY 1976

CENTER FOR RESEARCH IN SECURITY PRICES

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VOLUME 21, No. 1 May 13-14, 1976

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I. INTRODUCTION

The Black-Scholes formula value for a call option on a stock which pays no dividends during the life of the call, depends on just two parameters. They are "the" riskless rate of interest r and the volatility v of the underlying common stock. Uncertainties in either r or v lead to uncertainties in the calculated call option price.

We wish to determine r and v for the option formula in such a way as to either (a) improve the fit of the model prices to observed past and future option prices, or (b) to make the formula prices a more effective tool for capturing excess returns in the market place. This corresponds to the classification of models as (1) descriptive (i.e. fits data), (2) predictive (i.e. models based on deduction from principles, guided by the data and designed not only to fit the past but to forecast (fit) the future), and (3) prescriptive or normative (models which achieve certain criteria or objectives). See Thorp [27] and Morgenstern [24].

The Black-Scholes model is normative: the call option price $w(x,t)$ is a C^2 (continuous second partial derivatives) function such that (under their assumptions) a hedge, continously adjusted so as to be riskless, will yield the riskless rate of return. Black and Scholes [5] then test the formula to see if it is also (1)(from which we expect it has a good chance to be (2)).

Note that the model could be (3), but not (2) or (1), and yet be (b), providing a tool for excess returns. Hence (a) or (b) can each be independently true or false. These observations are critical for the use of the theory to gain excess returns in the market. Yet they are usually ignored in the literature. For example, the (normative) theory says a very deep in the money option should have a premium just slightly greater than

$$c(1 - \exp[r(t-t^*)]) \doteq cr(t^*-t),$$

the interest on c, where c is the option strike price, t is the time now and $t^* > t$ is the expiration time. An investor who buys 100 common and writes one such option (adjusting the hedge as necessary) gains the riskless rate on his net investment of $c \exp[r(t-t^*)]$. The market often "misprices" such options. If the premium is too large (as it occasionally is), the investor uses the preceding hedge to lock in a riskless rate in excess of that generally available.

More frequently, the premium is too small. Then firms who are able to use the common stock short sale proceeds can in effect borrow at below the riskless rate by selling 100 common and buying one call. Or, any investor who wants to purchase the stock can substitute the option. Then he has in effect purchased the stock plus the riskless excess return hedge. If the option premium is below the formula price by more than the excess transactions costs (if any), then he has added a net excess riskless return to his portfolio.

Note: This example is meant to illustrate how the option model can be normative and be a tool for providing excess returns, even though actual market prices differ substantially and systematically from model prices. However, there are other considerations (e.g. income taxes; see Thorp [28], Scholes [25]) as well as other option models, which singly or jointly could eliminate much or even most of this "mispricing".

We will be concerned below mainly with the use of stock volatility in options models. However we first discuss uncertainties in r, before putting them aside.

II. UNCERTAINTY OF THE RISKLESS RATE r

The appropriate "riskless" rate is that on debt which matures on the option expiration date [Merton, 21]. Uncertainty in this rate arises first from the fact that there is a rate r_1 at which money can be lent by the investor, and a generally different (greater) rate r_2 at which the investor can borrow. Which of these rates, if either, is appropriate? The call writer can get the riskless return r by buying stock long and continuously and appropriately adjusting the ratio of calls to stock. It can be shown that he has a net investment so he is in effect a lender of money, hence for him $r = r_1$ seems appropriate.

Similarly the buyer of calls can hedge with short stock. If the buyer can also use the short sale proceeds from the stock (e.g. a member firm with lendable stock), the buyer is a net borrower and should borrow at option model rate $r = r_2$. Now the formula price for the call option increases as r increases. Therefore if $r_2 > r_1$, then in the absence of transactions costs, the model price at which call options should be bought is greater than the price at which they should be written.

If such circumstances prevailed, the options markets would exhibit extraordinary volume. In the real world, these circumstances do not prevail. Most investors have significant transactions costs. The call option market price required to provide a given borrowing rate r_2 for the (hedged) buyer decreases as transactions costs increase. Similarly to provide a given lending rate r_1 to the (hedged) call option writer, the market price of the call option must increase as transactions costs increase. Thus, even though $r_2 > r_1$, if transactions costs are sufficiently high the formula's market price for the buyer will be less than that for the seller and no transactions should occur.

Even with sufficiently low transactions costs (negotiated commissions, or better yet, member firms and floor traders for their own accounts) volume will be limited by margin regulations for customers and net capital requirements and "haircuts" for members, applied to their finite available capital. Also, the options trading of many institutional investors is limited by a variety of regulations.

To find the appropriate r for calculating the call price, we are restricting consideration to "riskless" loans which mature when the option expires. Even so, there is not a single r_1 or a single r_2. The lender can, for instance, generally choose, among others, from T-bills, commercial paper, short term government agency paper, CDs and repo agreements. These rates generally differ.

Even if $r = r_1 = r_2$ for all loans and for all investors, before tax, it is the after tax return on investment that counts. (See Scholes [25] for a discussion of how taxes affect the option formula.) Denote the after tax rates by r_1^* and r_2^*. Then $r_1^* \leq r_1$, $r_2^* \leq r_2$, and the values of r_1^* and r_2^* will depend on the investor's tax situation. These values will vary widely from investor to investor and will vary for the same investor from one tax year to another.

There is reason to believe that the usual (before tax, nominal) indicators of r are too low. Black points out that the CAPM applied to observed risky rates implies a substantially larger r. He suggests that a larger r may be correct and that lenders may accept less than r in return for desired liquidity when they buy near money instruments like T-bills and commercial paper.

Lastly, it would seem that it is the real (rather than the nominal) after-tax rate $\tilde{r}$ that counts. In real terms, the usual indicators of nominal r are risky. Should the distinction between $\tilde{r}$ and r be taken into account by an options model?

Surprisingly, the variety of uncertainties in the value of r do not seem to cause serious difficulties in the use of the options formula as a tool for gaining excess returns. The explanation seems to be that uncertainties in the value of r affect the formula price very little for out of the money options, "moderately" for options around the strike price, and most for deep in the money options. But in the deep in the money case an investor can compute his own r_1 or r_2 from the market prices and transactions costs. Then he can virtually "lock it in" with the simple hedge previously described.

From the point of view of the profit seeker (nominal, pre-tax), deviations between market price and model price are simply welcome opportunities, in the deep in the money case. Thus when the model price is most affected by r uncertainties, the investor's practical use of the model is least affected. (The theoretician will, nonetheless, wonder about differences between model and market in the deep in the money region. Are they due to inefficiencies in the market-place? If so, how does one explain the existence/persistence of these inefficiencies? Or, is the model misspecified, and is there a (rational) model which significantly reduces the market-model price differences in this region?)

The investor is more concerned with uncertainties in calculated option model prices in the region around strike and below. As the stock price decreases, relative to the strike price, uncertainties in the volatility have an increasingly dominant effect on the calculated call prices. Thus, for the practical profit seeker, it appears that uncertainties in v are of greater concern than those in r.

We next consider uncertainties in the volatility.

III. DETERMINING THE VOLATILITY FOR THE SIMPLE BLACK-SCHOLES MODEL

The simple Black-Scholes model means here the formula for call options

$$w(x,t) = xN(d_1) - c\ \exp[r(t-t^*)]\ N(d_2) \qquad (1)$$

where x is stock price, c is the call exercise price, w is the call price, r is "the" riskless rate, and

$$d_{1,2} = [\ell n(x/c) + (r \pm v^2/2)\ (t^*-t)]\ v\ \sqrt{t^*-t}\ .$$

In deriving the formula, the stock price is assumed to be a stationary geometric Brownian motion (i.e. the lognormal model), i.e. for any t_1 and t_2, with $t_1 < t_2$, $\log\left(x(t_2)/x(t_1)\right)$ is $N(\mu,\sigma^2)$ (normally) distributed where $\mu = m(t_2-t_1)$ and $\sigma^2 = v^2(t_2-t_1)$. "Stationarity" refers to the assumption that m and v are constants, called the drift (or trend) and the volatility. Since m does not appear in the option formula (one of its remarkable features) we are only concerned with the effect of uncertainties in v.

Remark. The effect on w of the previously discussed uncertainty in r is $\Delta w \doteq (\partial w/\partial r)\ \Delta r$ and, in terms of the relative change in w, it is $\Delta w/w \doteq (\partial w/\partial r)\ \Delta r/w$, where w and $\partial w/\partial r$ come from formula (1). A calculation shows

$$\partial w/\partial r = (t^*-t)\ c\ \exp\Big(r(t-t^*)\Big)\ N(d_2) \qquad (2)$$

hence $\partial w/\partial r$ and hence Δw (to good approximation, for small Δr) is an increasing function of x, consistent with our earlier qualitative discussion.

On the other hand, $\Delta w \doteq (\partial w/\partial v)\ \Delta v$ for small changes in v, with the other variables held fixed. Using (1) again gives

$$\begin{aligned} \partial w/\partial v &\doteq \sqrt{t^*-t}\ c\ \exp\Big(r(t-t^*)\Big)\ N'(d_2) \qquad (3) \\ &= \sqrt{t^*-t}\ c\ \exp\Big(r(t-t^*)\Big)\exp\Big(-d_2^2/2\Big)(2\pi)^{-\frac{1}{2}} . \end{aligned}$$

This is greatest when $d_2 = 0$, which is equivalent to $x = c\ \exp[(v^2/2-r)(t^*-t)]$ or $x \doteq c$. Thus volatility changes cause the greatest changes in the formula option price for options around the strike price. It further turns out that the greatest relative change, given approximately by

$$\Delta w/w \doteq (\partial w/\partial v)\ \Delta v/w\ , \qquad (4)$$

tends to occur around the strike price or somewhat below it (out of the money).

To estimate v in the model, we first assume for simplicity that the drift parameter $m = 0$. Later we will remove this assumption. Then if $Y(t) = \ln X(t)$,

$$\Delta_i Y(t) \equiv Y(t_{i+1}) - Y(t_i) = \ln\Big(X(t_{i+1})/X(t_i)\Big) = vZ_i\ \sqrt{\Delta_i t}$$

where $\Delta_i t = t_{i+1} - t_i$ and the Z_i are normally $(0,1)$ distributed and independent, $i = 0,1,\cdots,n-1$. We wish to estimate v^2 from the observations $\{Y(t_i): i = 0,\cdots,n\}$.

Now $(\Delta_i Y)^2 = v^2 Z_i^2 \Delta_i t$ and $E\left((\Delta_i Y)^2\right) = v^2 E\left(Z_i^2\right) \Delta_i t = v^2 \Delta_i t$ so for the statistic $\sum_{i=1}^{n}(\Delta_i Y)^2$ we have

$$E\left(\sum_{i=1}^{n}(\Delta_i Y)^2\right) = v^2 \Sigma \Delta_i t = v^2(t_n - t_o) = v^2 \Delta t$$

where $\Delta t \equiv t_n - t_o$. Thus $\frac{1}{\Delta t}\sum_{i=1}^{n}(\Delta_i Y)^2 = \mathcal{E}_1$ is an unbiased estimator of v^2. The error is given by

$$\frac{1}{\Delta t}\sum_{i=1}^{n}(\Delta_i Y)^2 - v^2 = \frac{1}{\Delta t} v^2 \Sigma Z_i^2 \Delta_i t - v^2 = v^2\left(\Sigma Z_i^2 \frac{\Delta_i t}{\Delta t} - 1\right)$$
$$= v^2 \Sigma\left(Z_i^2 - 1\right) \frac{\Delta_i t}{\Delta t} = v^2 \sum_{i=1}^{n}\left(Z_i^2 - 1\right) p_i$$

where $p_i > 0$, $\Sigma p_i = 1$ is a set of weights.

We would like to "minimize" the error (in some sense) and the question arises as to whether these weights are optimal. If the p_i are very unequal the distribution of $\sum_{i=1}^{n}\left(Z_i^2 - 1\right)p_i$ will not be very "tight" and the estimate will be poor. Intuition suggests that equal weights, i.e. $p_i = 1/n$, would be "best."

Let "best" mean the set of p_i which minimizes the variance of the error, i.e. which give the most efficient estimator. Then we seek $\{p_i\}$ which minimize

$$f(p_1,\cdots,p_n) = E\left\{\left(\sum_{i=1}^{n}(Z_i^2-1)p_i\right)^2\right\} - \left\{E\left(\sum_{i=1}^{n}(Z_i^2-1)p_i\right)\right\}^2 .$$

Noting the last term is zero, and using the facts $E(Z_i^2-1) = 0$ and that the Z_i, hence the $(Z_i^2-1)p_i$, are independent, yields

$$f(p_1,\cdots,p_n) = \sum_{i=1}^{n} p_i^2 \left(E(Z_i^2-1)^2\right). \quad \text{Now}$$

$$E\left((Z_i^2-1)^2\right) = E(Z_i^4) - 2E(Z_i^2) + 1 = 3 - 2 + 1 = 2$$

so $f(p_1,\cdots,p_n) = 2 \sum_{i=1}^{n} p_i^2$ and the problem is to minimize $\sum_{i=1}^{n} p_i^2$ subject to the constraints $p_i \geq 0$, $\Sigma\, p_i = 1$. The solution (unique) is $p_i = 1/n$.

Thus the variance of $\mathcal{E}_1$ is minimized for $n + 1$ observations if and only if the $\Delta_i t$ are all equal. However the available "daily" data generally does not give equal $\Delta_i t$. This suggests that we can improve the estimate of v^2 by revising the estimator so that the $\Delta_i Y$ are equally weighted. Consider

$$(\Delta_i Y)^2/\Delta_i t = v^2\, Z_i^2\, . \tag{5}$$

Then for the estimator

$$\mathcal{E}_2 = \frac{1}{n} \sum_{i=1}^{n} \frac{(\Delta_i Y)^2}{\Delta_i t} = v^2\, \frac{1}{n} \sum_{i=1}^{n} Z_i^2\, , \tag{6}$$

$E(\mathcal{E}_2) = v^2$ so $\mathcal{E}_2$ is unbiased. The error is $\mathcal{E}_2 - v^2 = v^2\, \frac{1}{n}\, \Sigma\, Z_i^2 - v^2 = v^2 \sum_{i=1}^{n} (Z_i^2-1)/n$ and since the weights $p_i = 1/n$, this has minimum variance in the class of estimators of type $\mathcal{E}_1$. How does $\mathcal{E}_2$ compare to $\mathcal{E}_1$? The ratio

$$\operatorname{var}(\mathcal{E}_1)/\operatorname{var}(\mathcal{E}_2) = 2\Sigma\, p_i^2/2\Sigma\,(1/n)^2 = n\,\Sigma\, p_i^2.$$

But $\Sigma\left(p_i - \frac{1}{n}\right)^2 = \Sigma p_i^2 - 1/n$ whence $n \Sigma p_i^2 = 1 + n \Sigma \left(p_i - \frac{1}{n}\right)^2 = 1 + n \operatorname{var}(P)$ where P is a random variable uniformly distributed over $p_1, \cdots, p_n$.

In the calculation of stock volatility, using one year's "daily" data, the effect is negligible: take 52 weeks with $\Delta_i t = 1$ day for 4×52 times and $\Delta_i t = 3$ days for 1×52 times. Then $n = 260$ and $\Delta t = 364$ days, from which

$$n \operatorname{var}(P) = 260[(1/364-1/260)^2 + (3/364-1/260)^2]$$
$$= .005330$$

so $\sigma^2(\mathcal{E}_1)/\sigma^2(\mathcal{E}_2) = 1.005330$ whence $\sigma(\mathcal{E}_1)/\sigma(\mathcal{E}_2) = 1.00266$, a difference less than 0.3%.

IV. SAMPLING (MEASUREMENT) ERRORS FOR v^2

For weekly and monthly data, $\Delta_i t \doteq 1/n$ so the distinction between $\mathcal{E}_1$ and $\mathcal{E}_2$ disappears. But daily data gives us sharper estimates of v^2. This suggests the next question: How accurate is our estimate of v^2?

<u>Case 1: m = 0</u>

From (5) and (6), $\mathcal{E}_2 = (v^2/n)\,\chi^2$ where χ^2 is Chi square with n degrees of freedom. Calculations show that a 90% confidence interval for one year's (equally spaced) weekly data (52 degrees of freedom) is given by

$$\chi^2_{.05} < 52\, e_2/v^2 < \chi^2_{.95} \qquad (7)$$

where e_2 is the observed value of $\mathcal{E}_2$. In terms of v, the 90% confidence interval becomes

$$.86\ \sqrt{e_2} \leqq v \leqq 1.20\ \sqrt{e_2}\ . \tag{8}$$

If the year is divided into 250 equal time intervals by observations (an approximation to daily data) the corresponding 90% confidence interval is

$$.93\ \sqrt{e_2} \leqq v \leqq 1.08\ \sqrt{e_2}\ .$$

Case 2: m Arbitrary Known Constant

Proceeding as before, $\Delta_i Y(t) = v\, Z_i \sqrt{\Delta_i t} + m\, \Delta_i t$ where the Z_i are normally $(0,1)$ distributed and independent, $i = 0,1,\cdots,n-1$. Corresponding to (6), we obtain

$$\mathcal{E}_3 = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\Delta_i Y - m\Delta_i t}{\sqrt{\Delta_i t}} \right)^2 = v^2\, \frac{1}{n} \sum_{i=1}^{n} Z_i^2 = (v^2/n)\ \chi^2\ , \tag{9}$$

with the same confidence limits as (7) and (8). The usefulness of this Case 2, which appears to provide little that is different from its particular subcase, Case 1, is that it shows (a) that the value of m has little effect on the estimated value of v and (b) it suggests to us how to estimate v when m is unknown (Case 3, below).

To verify (a),

$$E(\mathcal{E}_3) = \frac{1}{n} \sum_{i=1}^{n} [(\Delta_i Y)^2/\Delta_i t)] - \frac{1}{n} \sum_{i=1}^{n} m^2\ \Delta_i t = E(\mathcal{E}_2) - m^2/n\ , \tag{10}$$

assuming as before one year's data, i.e. $\Sigma\, \Delta_i t = 1$.

Now $\mathcal{E}_3$ is unbiased therefore if we neglect m (i.e. set $m = 0$)

and use $\mathcal{E}_2$ to estimate v^2, the bias is m^2/n. For the geometric Brownian motion which the stock price $X(t)$ is assumed to follow, we define R by

$$\exp[R(t_2-t_1)] = E(X(t_2)/X(t_1)) = \exp[(m+v^2/2)(t_2-t_1)]$$

or $R = m + v^2/2$. It is the "mean compound growth rate" or the "exponential growth rate."

The Fisher and Lorie work indicates a long term average value of about 0.083 for R. Observed v values of stocks underlying listed options have generally been bracketed between 0.2 and 0.7 with the median near 0.35. Using $R = 0.083$, the m values corresponding to $v = 0.2$, 0.35 and 0.7 respectively are 0.063, 0.022, and -0.162, their squares are 0.0040, 0.0005, and 0.0262, and the corresponding v^2 values are 0.0400, 0.1225, and 0.4900. The percent positive bias, $100\ m^2/nv^2$, from neglecting m in the estimate of v is shown in Table 1. Note that the bias is very small.

One expects riskier (higher v) stocks to have higher mean compound returns R. This leads to less extreme values for the bias. Table 2 shows the results if we change the Table 1 assumption to: $R = .06$ for $v = 0.20$, $R = .083$ for $v = 0.35$, and $R = .12$ for $v = 0.70$. We suspect that most users of the Black-Scholes formula use some weighted version of $\mathcal{E}_2$, neglecting m and obtaining v^2 values that are slightly too large. The errors thus introduced appear negligible. We shall, therefore, omit discussion of the actual situation, Case 3: mean unknown, since Case 1 appears to approximate it very well indeed.

Since $\Delta_i Y = v\ Z_i \sqrt{\Delta_i t} + m\Delta_i t$, in the case $\Delta_i t = 1/n$ for all i, $\Delta_i Y = v\ Z_i/\sqrt{n} + m/n$. Thus $\Delta_i Y$ is normally μ,σ^2 distributed for each i,

TABLE 1

Bias m^2/v^2 in the Estimate of v^2 when m is Assumed to be Zero, using $0.083 = R = m + v^2/2$ to Estimate m

v	v^2	m	m^2	m^2/v^2	$m^2/50v^2$
0.20	.0400	.0630	.0040	10.0%	0.2%
0.35	.1225	.0217	.0005	0.4%	0.0%
0.70	.4900	-.1620	.0262	5.3%	0.1%

TABLE 2

Bias m^2/v^2 under the Changed Assumptions
$R = 0.06$ if $v = 0.20$, $R = 0.12$ if $v = 0.70$

v	v^2	m	m^2	m^2/v^2	$m^2/50v^2$
0.20	.0400	.0400	.0016	4.0%	0.1%
0.35	.1225	.0217	.0005	0.4%	0.0%
0.70	.4900	-.1250	.0156	3.2%	0.1%

where $\mu = m/n$ and $\sigma^2 = v^2/n$. Letting $\overline{\Delta_i Y} = \frac{1}{n} \sum_{i=1}^{n} \Delta_i Y = \left(Y_n - Y_o\right)/n$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(\Delta_i Y - \overline{\Delta_i Y}\right)^2$, it is a standard result that $n\,S^2/\sigma^2$ is $\chi^2(n-1)$ distributed. Now $n\,S^2/\sigma^2 = \frac{n^2}{(n-1)v^2} \sum_{i=1}^{n} \left(\Delta_i Y - \overline{\Delta_i Y}\right)^2$ so we can use this statistic to get confidence intervals for v^2 when m is unknown and all $\Delta_i t = 1/n$.

V. BIASED OPTION FORMULA PRICES COME FROM UNBIASED v^2

Suppose we have an unbiased estimator of v^2, such as $\mathcal{E}_2$ in case $m = 0$. Then it is well known that a function $h(\mathcal{E}_2)$ will in general be a biased estimator of $h(v^2)$. (In particular $\sqrt{\mathcal{E}_2}$ is a biased estimator of $\sqrt{v^2} = v$.) Thus, considering $w(x,t)$ from equation (1) as a function $h(v^2)$, $h(\mathcal{E}_2)$ is likely to be a biased estimator of $h(v^2)$.

It seems desirable to use the data $\{x(t_i): i=0,\cdots,n\}$ in such a way as to obtain an unbiased estimate of $w(x,t)$. Let W be such an unbiased estimator of $w(x,t)$. Then $E(W) = w(x,t)$ is the option formula value $w(x,t,v)$ for the true (but unknown) v. I have not finished thinking about this problem but a choice for W which seems plausible to me is:

$$W = \frac{1}{n} \sum_{i=1}^{n} w(x,t,(\Delta_i Y)^2/\Delta_i t) \tag{11}$$

where $w(x,t,v^2) \equiv w(x,t)$ as given by (1).

Thus the practitioner might compute and use as his estimate of $w(x,t)$

$$\overline{w(x,t)} = \frac{1}{n} \sum_{i=1}^{n} w(x,t,(\Delta_i Y)^2/\Delta_i t) \;. \tag{12}$$

If (11) is appropriate, how would the calculated model prices change? The numbers $\overline{w(x,t)}$ in (12) are an average of $w(x,t,v_i^2)$ values for v_i^2 ranging over values both above and below the value $v^2 = \frac{1}{n} \sum_{i=1}^{n} (\Delta_i Y)^2/\Delta_i t$. Since $w(x,t,v^2)$ is increasing in v^2, values $v_i^2 > v^2$ will tend to make $\overline{w(x,t)} > w(x,t,v^2)$ and values $v_i^2 < v^2$ will tend to make $\overline{w(x,t)} < w(x,t,v^2)$.

Experience with the formula suggests that $\overline{w(x,t)} > w(x,t)$ in the regions of interest and that the most significant increase in the calculated option prices is for out of the money options. (Note that $\overline{w(x,t)} < w(x,t)$ is possible. Suppose, for example, that $v_1^2 = \cdots = v_{n-1}^2 = 0$ and $v_n^2 = n\, v^2$, $r = .1$, $t^*-t = .5$, $x = 200$, $c = 100$, $v^2 = .1$. Then $w(x,t,v_i^2) = 200 - 100\, e^{-.05}$, $i = 1,\cdots,n-1$, and $w(x,t,v_n^2) < 200$ so $\overline{w(x,t,v^2)} < 200 - 100\, e^{-.05} + 100\, e^{-.05}/n$ $\rightarrow 200 - 100\, e^{-.05} < w(x,t,v^2)$ hence for sufficiently large n, $\bar{w} < w$.) This is also a region where the model values are generally significantly below market prices. Hence removing the bias in w may remove much or all of this difference.

We hope to find out whether W is unbiased for w and, if so, will calculate model prices $\bar{w}$ and w for a variety of situations, to investigate whether $\bar{w}$ is significantly greater than w.

Here is a method which could tell us to what extent the calculated model value of w, based on an estimated v, differs from the true model value, based on the true v. Assume for simplicity that $m = 0$, which is Case 1 of section 4. Then the data-based estimator $\mathcal{E}_2 = v^2/n \sum_{i=1}^{n} Z_i^2 = (v^2/n)\, \chi^2(n)$ where $\chi^2(n)$ is Chi-square with n degrees of freedom. Ordinarily the observed value of e_2 is substituted for v^2 in the Black-Scholes formula, giving an option value $w(e_2)$.

Now let $f(e_2)$ be the probability density function for e_2, which we noted is $\chi^2(n)$. Then

$$\int w(e_2)\ f(e_2)\ de_2 = E(w(\mathcal{E}_2))$$

is the expectation of the calculated model value of w, based on v estimated from the data. We can calculate $E(w(\mathcal{E}_2))$ and $w(v^2)$ for various true v and assumed r. This can be carried out theoretically or, if necessary, numerically.

As an extreme illustration of what might be learned, suppose it turns out that $E(w(\mathcal{E}_2)) < w(v^2)$ but that $E(w(a\mathcal{E}_2)) \doteq w(v^2)$ where $a > 1$ is some constant. Then ae_2 should be used in the Black-Scholes formula for v^2, rather than e_2. Further, to use e_2 as is customary, would give option model prices which are biased downward, especially for out of the money options (as seems to be the case).

VI. THE BLACK-SCHOLES (v^2x^2) MODEL AND THE COX-ROSS (v^2x^θ) MODELS MAY BE CONSISTENT

There is a rapidly expanding literature on options models alternate to the Black-Scholes model (see references). It is not yet known whether any of these alternate models work better than the original, either in making predictions of market prices or as tools for securing excess rates of return.

These options models are based on various assumptions about the underlying stochastic behavior of the stock. A most important parameter here is "instantaneous volatility," ν, which we define when it exists by $\nu^2 = \lim_{\Delta t \to 0} E((\Delta x)^2/\Delta t) = v^2\ f(x)$, with v constant. Note that ν may depend on x but not on t. For the Black-Scholes model $f(x) = x^2$ and $\nu = vx$. However, it has long been suggested that $\nu = vx^\alpha$, $0 < \alpha < 1$, better describes observed stock price changes. I first read of this in 1964 in Burton Crane's "The Sophisticated Investor," where he claimed $\nu = v\sqrt{x}$. It seemed to be a cross-sectional assertion. Hence, even if the "average" $\bar{\alpha}$ were 1/2, the α_i for various stocks might differ significantly from 1/2. I did a small rough cross-sectional study at the time and concluded that $\bar{\alpha}$ was nearer to 0.85 than to 0.5.

As a possible explanation, consider the various plausibility arguments for the lognormal description of stock prices. These arguments seem still more plausible when applied to the total assets (debt + equity) of the company. Suppose then that the price of the total assets of a company follows the lognormal model. What model describes the price behavior of the common stock? The following example supports the idea that $\nu = vx^\alpha$, $0 < \alpha < 1$, is approximately correct. We oversimplify to illustrate the point; many of the restrictions can be lifted.

Imagine a company which has common stock and a single bond issue, with no other debt (even bank loans) of any kind. To simplify the coming use of the Black-Scholes model, assume further that the stock pays no dividends and the bond has no coupon (to be more nearly realistic, think of it as having been issued at an appropriate discount). Suppose that both stock and bonds are continuously and actively traded on an exchange so we have the usual price information. Let $b(t)$ be the market price of the entire bond issue, $w(t)$ the market price of all the stock, and c the face amount of the entire bond issue.

Let $x(t) = w(t) + b(t)$ and note that this is the current market price of the total assets of the company. Call a claim on the total assets "superstock." If we think of stock and bonds as divided into N small shares each, which we can pair to make N small shares of superstock, then the market prices of stock and bonds give an effective market price for supershares. Note that x/N, w/N and b/N are marginal prices: we can buy small quantities at these prices but if we try to buy a whole issue, or the whole company, the price will probably rise and we will pay more than w, b, or x. Thus "market price" for an issue here means (price/share) x (no. of shares), rather than what it would actually cost to buy the issue.

When the bond issue matures at time t^*, we expect the stockholders to redeem the bonds if the company assets have market price greater than c, i.e. if $x(t^*) > c$. Upon redemption the stock issue should have market price $w(t^*) = x(t^*) - c$. If instead $x(t^*) \leq c$ at t^*, the stockholders should let their option to redeem the bonds expire unexercised. The company defaults and $w(t^*) = 0$.

It follows then that in this simplified example the assumptions of the Black-Scholes option model are fulfilled. The stock is a (Euro) call option on the superstock with exercise price c and expiration date t^*. (The idea up to this point appears in Black and Scholes [6]. See Smith [26], §7.1, for a compact account.) Using the notation and methodology of the appendix, we have

$$\Delta x = v\, x\, Z \sqrt{\Delta t} + O(\Delta t) \quad \text{and} \quad \Delta w = w_1 \Delta x + O(\Delta t)$$

hence the ratio of the relative change in equity of the stock to the superstock is

$$v_w/v_x \doteq \frac{\Delta w/w}{\Delta x/x} = \frac{[w_1\ v\ x\ Z \sqrt{\Delta t} + O(\Delta t)]/w}{[v\ x\ Z \sqrt{\Delta t} + O(\Delta t)]/x} \doteq \frac{w_1 x}{w} \qquad (13)$$

where we put $v_w = \nu_w/w$ and $v_x = \nu_x/x = v$, whence v_x and v_w are the respective "lognormal variances." Suppose now that $w_1 x/w = w^{-\gamma}$, $0 < \gamma < 1$. Then

$$\Delta w \doteq w_1 \Delta x = (w_1 x/w)\ w\ v\ Z\ \sqrt{\Delta t} = w^{1-\gamma} v\ Z\ \sqrt{\Delta t}$$

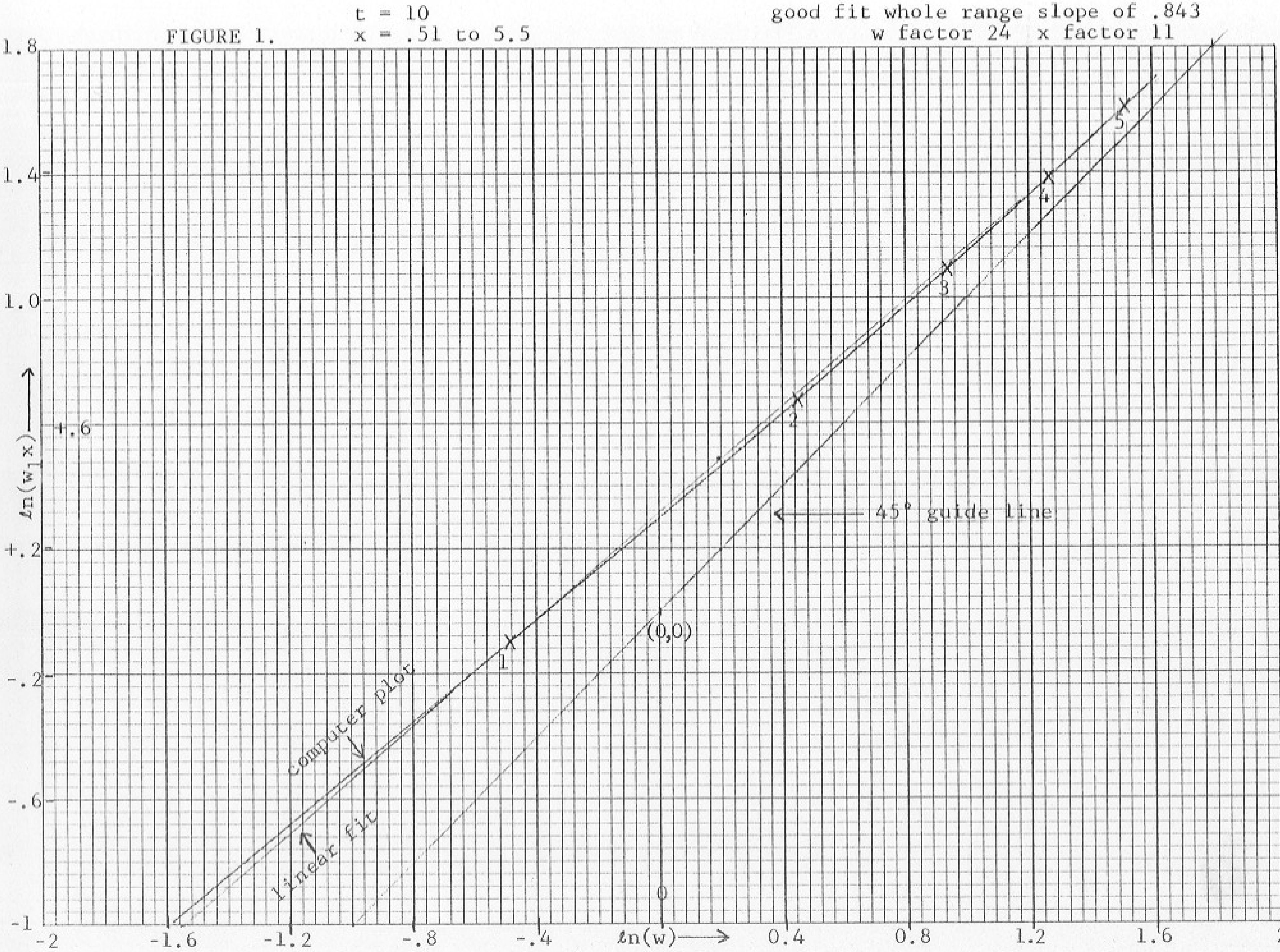
$$= w^{\alpha} v\ Z\ \sqrt{\Delta t}\quad ,$$

where $\alpha = 1 - \gamma$, $0 < \alpha < 1$, and w diffuses according to a Cox-Ross [9] "constant elasticity of variance" model.

The expression $w_1 x/w = v_w/v_x = v_w/v$ has the right qualitative behavior. From (1), calculations yield $w_1 x/w = 1 + c \exp[-r(t^*-t)]\ N(d_2)/w$ and it can be shown from this that $w_1 x/w \to \infty$ as $x \to 0$.

We have $w_1 x = w^{\alpha}$ where α is a function $\alpha(x)$, $0 < \alpha(x) \leq 1$. For lognormality and the Black-Scholes model $\alpha(x) = 1$ and the quantity $1 - \alpha(x)$ is an indication of the extent of deviation of the stock model from lognormality. Taking logarithms, $\alpha(x) = \log(w_1 x)/\log w$. If $\alpha(x) \doteq \alpha_o$ over an x interval then the Cox-Ross model is approximately true for that α_o and that interval.

Figure 1 is a computer plot of $\alpha(x)$ when $t^* - t = 10$ years, $v = 0.3$, $r = 0.08$, and $c = 1$. The curve was drawn by calculating $(\ell n\ w(x),\ \ell n(w_1 x))$ for $x = .51(.01)5.50$ and plotting this parametrically described (by x) curve, joining successive computed points. The points for integer values of x are marked by labelled crosses. Superimposed is a lighter straight line. This is a "best" linear fit estimated by eye. Note that it is a remarkably good fit over the entire range of the plot. The slope of this line is $6.6/7.77 = 0.849$. It is curious that it happens to coincide so exactly with the 0.85 earlier reported. The range of x values covered is from 0.51 to 5.5, a ratio of superstock prices of more than 10. The range of w values is from $w = e^{-1.6}$ to $w = e^{1.6}$, a ratio of stock prices of more than 24, a far wider range than normally encountered during the life of a listed call option.



t = 10
x = .51 to 5.5
good fit whole range slope of .843
w factor 24 x factor 11
1.8
1.4
1.0
+.6
+.2
-.2
-.6
-1
-2
-1.6
-1.2
-.8
-.4
0
0.4
0.8
1.2
1.6
2
$\ell n(w_1 x)$
$\ell n(w)$
(0,0)
45° guide line
computer plot
linear fit
1
2
3
4
5

FIGURE 1.

Since $v_w/v_x = w_1x/w$, and the graph gives (for the linear fit) $\ell n\, w_1x = .32 + .849\, \ell n\, w$, we find $\ell n(v_w) = .32 - .151\, \ell n\, w$ and $v_w = 1.38w^{-.151}v_x$ is the relation, derived from the linear fit, between the superstock volatility v_x and the stock volatility v_w.

Figure 2 is a similar graph, for $t^* - t = 5$ years. Again there is a remarkable linear fit over the entire range. The values of x plotted are $0.51(.01)5.21$. The slope is 0.766, the ratio of x values covered is more than 10, and the ratio of w values covered is $\exp(1.5 + 2.45) \doteq 52$. The graph gives for the linear fit $\ell n\, w_1x = .458 + .766\, \ell n\, w$ or $v_w = 1.58w^{-.234}v_x$.

We also have graphed the cases $t^* - t = 1$ year and $t^* - t = 1$ week. These are shown in Figures 3, 4 and 5. The range and precision of the linear fit decreases as $t^* - t$ decreases. However, the amount the superstock or the stock can change in the time $t^* - t$, also decreases as $t^* - t$ decreases. When this is considered, there is a very good linear fit about each point $(\ell n\, w(x), \ell n\, w_1x)$ on the curves. This example could be extended to companies with several classes of debt.

It would be interesting to see if there are any companies which approximately fit the assumptions: listed common stock, one class of listed debt, and no other securities or debts. Then the option model could be used as in the example to calculate common stock prices, for comparison with the actual market prices.

The example, and more generally the model of stocks as options on superstocks, suggests the principle that common stocks should tend to be more volatile as the debt-to-equity ratio increases. We can test this two ways: cross-sectionally, and with individual stocks. Cross-sectionally, one might plot volatility of a stock average versus

FIGURE 2. $t = 5$ $x = .51$ to 5.5

slope over range $.51 \leq x \leq 3$ is $.76$ with very good fit w factor 27 x factor 6

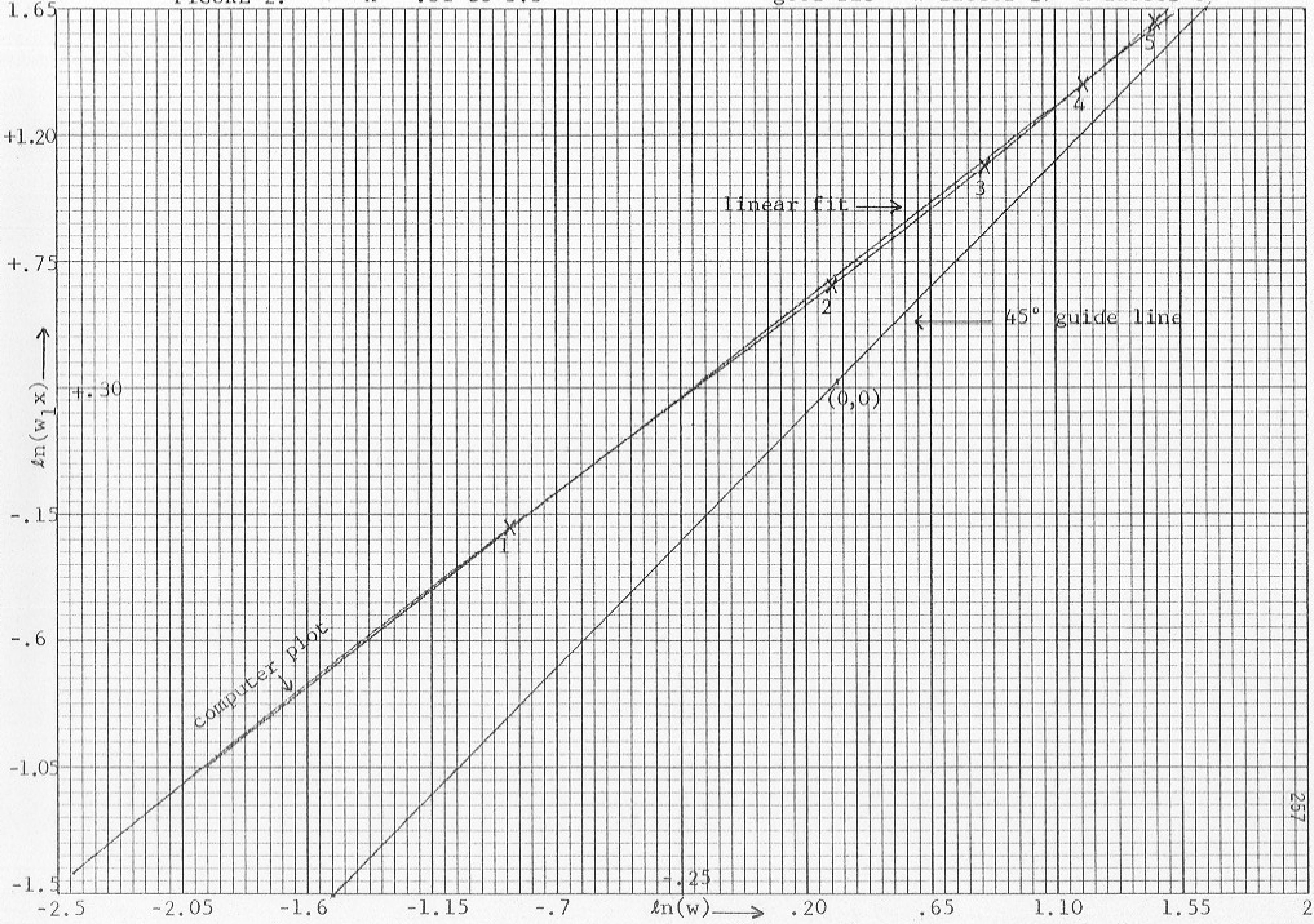


FIGURE 3. $t = 1$, $x = .31$ to 5.3

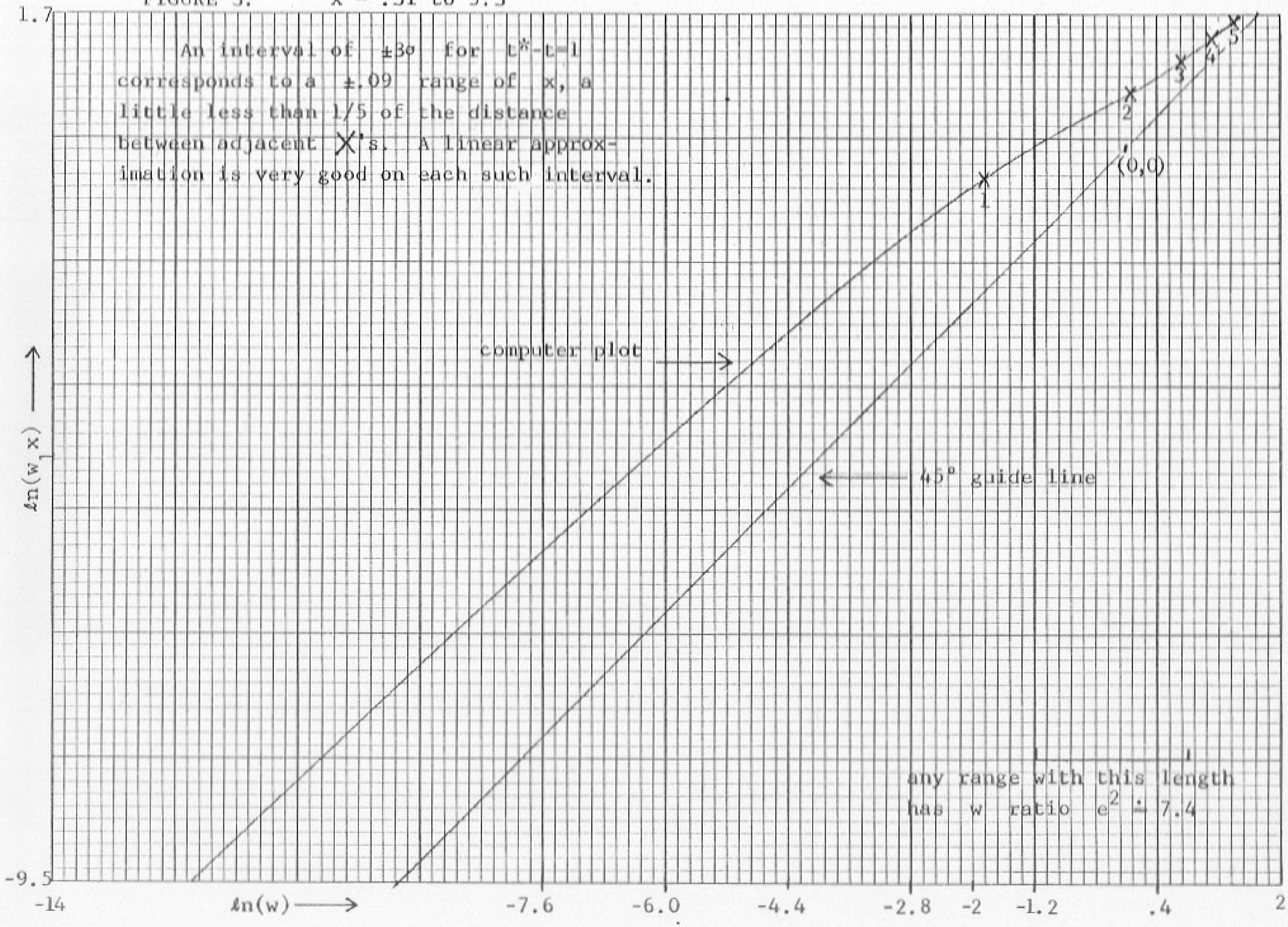


FIGURE 4. $t = .02$, $x = .91$ to 1.1 Step $.01$

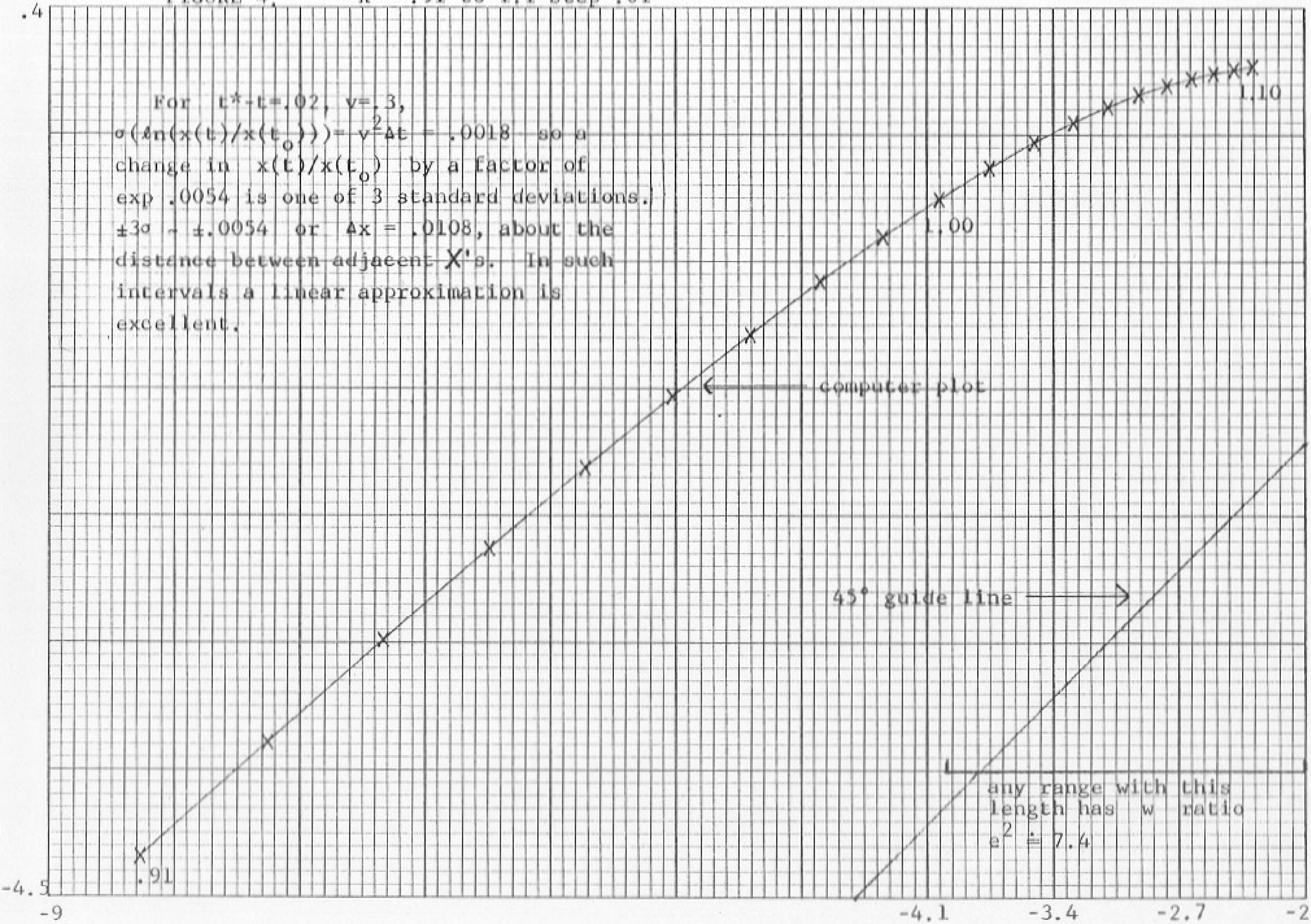
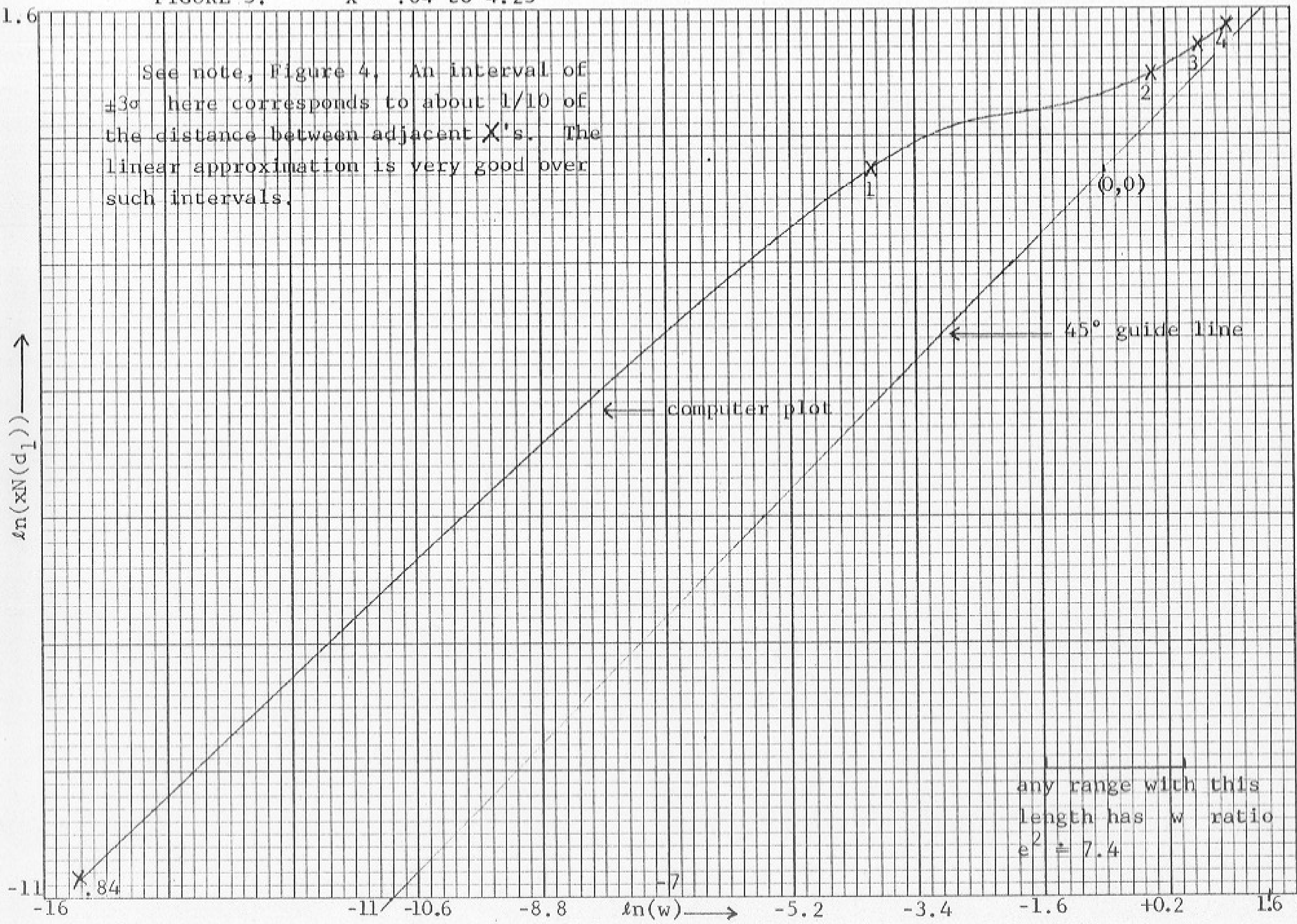


FIGURE 5. $t = .02$ $x = .84$ to 4.25



debt-to-equity ratio and look for a positive correlation. There has been a marked secular increase in debt-to-equity for American corporations in the post-war era. Has there been a significant increase in stock volatilities? According to Kidder, Peabody [17], in an interesting discussion of market volatility, there has been such an increase. They try various one-variable explanations: institutionalization of trading, interest rate fluctuations, and a trend (tax-related) towards lower dividend payouts ratios. None seems satisfactory. We suggest that they add a new variable, debt-to-equity ratio, and that they then use a multi-linear regression on the whole set of variables.

In shorter times (like a year) when the debt-to-equity ratio is determined mainly by fluctuations in stock prices, does the volatility vary like vx^{α}, $0 < \alpha < 1$, rather than vx? In other words, do percentage changes in the stock averages tend to be larger when prices have fallen, than when they have risen? (Note: Over longer periods, other factors may dominate effects of price change on volatility. For instance, over the post-World War II years, stock prices have risen greatly. This might be expected to reduce stock relative volatility. However the debt-to-equity ratio has also risen greatly with the effect of raising volatilities. The net effect seems to be a significant increase in volatilities.)

This notion of relative volatility increasing as stock prices decrease may explain several puzzles in the literature. Thorp and Kassouf [29] report a regression model for warrant prices, devised by Kassouf. The model is $w(x) = (1+x^z)^{1/z} - 1$ where z is a function of several variables that is determined by regression on data. In particular z is found to be increasing in $x/\bar{x}$, where $\bar{x}$ is a surrogate for the average stock price over the preceding year. Since increasing z means decreasing w, the regression says that if the stock price has risen, the model price is higher, but lower than otherwise. Conversely, if the stock price has fallen, the model price is lower, but higher than if $x/\bar{x}$ were not so small.

In terms of the Black-Scholes model, if stock price w increases and the volatility of w increases more slowly than w (as it does when w is an option on the superstock), the model price is less than otherwise expected. Conversely, if w falls, we expect the volatility to be higher than previously believed, thus a higher model price. My casual observations during a decade of trading warrants and options and watching their price action confirms this behavior. A quantitative verification would be desirable.

Note also that this effect: option (and warrant) prices lagging below model prices when the stock runs up, and lagging above model prices when the stock runs down, tends to "explain" at least part of the two principal discrepancies which seem to exist, between the Black-Scholes model and market prices.

Leabo and Rogalski [20] reported that warrant price changes do not seem to follow the lognormal distribution. They conclude that warrant price changes are not consistent with the random walk model. Our analysis shows that if the lognormal model holds for stocks and if the Black-Scholes model applies, then we should expect warrant price changes <u>not</u> to be lognormally distributed. A detailed analysis along these lines may explain away the Leabo-Rogalski results and show that warrant price changes <u>are</u> consistent with the lognormal model for stock price changes.

VII. TESTING THE COX-ROSS VOLATILITY MODEL

Assume that for small time increments $\Delta x = vx^{\beta} Z \sqrt{\Delta t}$ where v and β are positive constants characteristic of the stock. In particular the trend $m = 0$. Then $(\Delta_i x)^2 = v^2 x_i^{2\beta} Z_i^2 \Delta_i t$ and $u_i \equiv (\Delta_i x)^2 / x_i^{2\beta} \Delta_i t$, $i = 1,2,\cdots,n$, has a common distribution, namely $v^2 \chi^2(1)$. For various values of β regress u_i versus x_i and ask whether the slope b differs "significantly" from 0. When β is small the slope will be positive. When β is large, the slope will be negative. As β increases, the slope decreases monotonically.

There will be an interval in which b does not differ "significantly" from 0. Because of the monotone behavior of b, a binary search quickly determines this interval to arbitrary precision. This interval is a confidence interval for β.

To apply the procedure cross-sectionally, take the time interval $\Delta_i t = \Delta t$ fixed and let $\Delta_i x$, x_i refer to the ith stock. Interpret v and β as "average" values for the set of stocks. One might use, for instance, the x_i and $\Delta_i x$ for one particular week on the New York Stock Exchange.

VIII. CLOSING REMARKS

In the complete version of this paper I want to also discuss:

A. Use of past stock price data to determine volatility.
   1. Optimal weighting (near term versus more remote).
   2. Regression towards the mean. Cf. Klemkosky and Martin [18] for β.
   3. Bayesian estimates (ibid).

B. Using option volatility to (help) forecast volatility: Latané and Rendleman [19].

C. Alternate option models.
   1. Jump processes. Merton, Cox-Ross.
   2. Possible role of noise component in the stochastic process describing stock price. Fernholz [12].

3. "Vertical" vibration about option model surface; vertical drift.

D. Time variation in v.

1. May change via a random walk. Fisher [13], Black [1,2].
2. "Volume time" versus calendar time.

I wish to thank David Gelbaum for programming assistance and calculations, and Oakley, Sutton Securities Corporation for supporting this work.

APPENDIX

A Corrected Derivation of the Black-Scholes Option Model

INTRODUCTION

The derivation of the option model by Black and Scholes [6] depends upon stochastic calculus and Itô's lemma. This demands much of the reader. Furthermore the derivation as given contains some mathematically incorrect steps. The revised derivation which we give here appears to be more nearly mathematically correct, intuitive, relatively undemanding of reader background, and very close in form to the original.

THE REVISED DERIVATION

Notation, terminology, and procedure follow Black and Scholes [6], pp. 641-643 equations (1)-(7).

Create a hedged position with one share of stock long and

$$1/w_1(x,t) \tag{1}$$

options sold short. We introduce (implicitly) at this point the assumption that there is a continuous "smooth" (i.e. "sufficiently many" continous partial derivatives) function $w(\cdot,\cdot)$ such that the value of the option is $w(x,t)$ for stock price x and time t.

To a first approximation, the value of the hedged position does not depend on the price of the stock. To see this, expand Δw by Taylor's formula:

$$\Delta w = w_1 \Delta x + w_2 \Delta t + O(\Delta t) \tag{1a}$$

where we use the assumption that

$$\Delta x = x[\exp\{vZ\sqrt{\Delta t}+m\Delta t\} - 1] = mx\Delta t+vxZ\sqrt{\Delta t}+O(\Delta t) = vxZ\sqrt{\Delta t}+O(\Delta t) \quad (1b)$$

(stock price changes are lognormally distributed, i.e. they follow a geometric Brownian motion with drift). The constants m and v are called "drift" and "volatility" respectively. The Z represents a normally distributed random variable with zero mean and unit variance. This assumption (1b) corresponds to Black and Scholes' assumption (b). The symbol $O(y)$ means that for some $\varepsilon > 0$ and $M > 0$ we have $|O(y)| < M|y|$ if $|y| < \varepsilon$. We get (1a) from (1b) by noting that $\Delta x = O\left((\Delta t)^{\frac{1}{2}}\right)$ so all higher order terms in (1a) can be replaced by $O(\Delta t)$.

What is new here is the appearance of Z in equation (1b). It is the crucial difference. I am indebted to Bob Oliver for showing me that it was required. Note that without Z, (1b) would equate the random variable Δx to the constant $mx\Delta t + vx\sqrt{\Delta t} + O(\Delta t)$. This is evidently incorrect. It's subsequent role will be still more crucial.

We can write (1a) as

$$\Delta w = w_1\Delta x + O(\Delta t) . \quad (1c)$$

Thus if the stock price changes by Δx, the option price will change by $w_1\Delta x + O(\Delta t)$ and the number $1/w_1$ of options will change by $\Delta x + O(\Delta t)$ hence the change in the equity of the portfolio is $O(\Delta t)$.

With these revised preliminaries we derive the model.

The value of the equity is

$$E = x - w/w_1 . \quad (2)$$

The change in the value of the equity E in a short time Δt is:

$$\Delta E = \Delta x - \Delta w/w_1 \ . \tag{3}$$

Now for the crucial correction: Assuming the short position is changed continuously, we have from Taylor's formula and (1b) applied to higher order terms:

$$\Delta w = w_1\Delta x + w_2\Delta t + \frac{1}{2}(w_{11}(\Delta x)^2 + 2w_{12}\Delta x\Delta t + w_{22}(\Delta t)^2) + O\Big((\Delta t)^{3/2}\Big) \ .$$

Using (1b) and lumping all terms of order higher than Δt,

$$\Delta w = w_1\Delta x + w_2\Delta t + \frac{1}{2}w_{11}v^2x^2Z^2\Delta t + O\Big((\Delta t)^{3/2}\Big) \ . \tag{4}$$

Substituting in (3) yields

$$\Delta E = -\Big(\frac{1}{2}w_{11}v^2x^2Z^2 + w_2\Big)\Delta t/w_1 + O\Big((\Delta t)^{3/2}\Big) \ . \tag{5}$$

Note that in the original derivation the Z^2 is missing at this point. Hence the random variable ΔE is being incorrectly equated there to a constant. Black and Scholes in fact continue by saying: "Since the return on the equity in the hedged position is certain, the return must be equal to $r\Delta t$. . . . Thus . . .

$$-\Big(\frac{1}{2}w_{11}v^2x^2 + w_2\Big)\Delta t/w_1 = (x - w/w_1)r\Delta t \ ." \tag{6}$$

Returning to (5) we proceed differently. Note first that the Z^2 term shows that ΔE is risky. However, as the time intervals between adjustments in the hedge ratio tend to zero, we shall show that the risk in a fixed time interval Δt also tends to zero. Subdivide Δt into n equal subintervals.

$$\Delta_i t = [i\Delta t/n, (i+1)\Delta t/n], \quad i = 0, \cdots, n-1 .$$

Let $\Delta_i E$ be the change in equity corresponding to $\Delta_i t$ and note that

$$\begin{aligned} \Delta_i E &= -\left(\tfrac{1}{2}w_{11}v^2x^2Z_i^2 + w_2\right)\Delta_i t/w_1 + O\left((\Delta_i t)^{3/2}\right) \\ &= -\left(\tfrac{1}{2}w_{11}v^2x^2Z_i^2 + w_2\right)\Delta t/nw_1 + O\left((\Delta t/n)^{3/2}\right) \end{aligned} \tag{6a}$$

where the Z_i are independent $(0,1)$ normally distributed random variables. Then with n-step adjustment of the hedge,

$$\begin{aligned} \Delta E = \sum_{i=1}^{n} \Delta_i E = &- \tfrac{1}{2}w_{11}v^2x^2\left(\tfrac{1}{n}\sum_{i=1}^{n} Z_i^2\right)\Delta t/w_1 \\ &- w_2\Delta t/w_1 + O\left((\Delta t)^{3/2}\right)/\sqrt{n} . \end{aligned} \tag{6b}$$

By the law of large numbers,

$$\lim_{n\to\infty} \Delta E = -\tfrac{1}{2}w_{11}v^2x^2\Delta t/w_1 - w_2\Delta t/w_1 \tag{6c}$$

in probability.

Thus continuous adjustment of the hedge yields, with probability 1, for the small <u>finite</u> time Δt, the <u>riskless</u> change in equity of (6b). Equating this to $(x - w/w_1)r\Delta t$ yields equation (7) of Black and Scholes,

$$w_2 = rw - rxw_1 - \tfrac{1}{2}v^2x^2w_{11} \tag{7}$$

from which point the derivation continues as in their paper.

MATHEMATICAL REMARK

To preserve the intuitive flow of the argument we have omitted certain readily supplied but tedious steps: First, the expansions (1a), (1b), (1c), (4), (5), (6a), (6b), etc. are correct in some bounded region $|\Delta x| < \delta_1$, $|\Delta t| < \delta_2$, but $|\Delta x|$ is unbounded. However, by choosing sufficiently small δ_2 and given δ_1, $\Pr\{|\Delta x| \geqq \delta_1\}$ can be made as small as we like. Hence the expansions are correct except on a set of arbitrarily small probability. Secondly, we should observe that the Taylor expansion (5) is valid uniformly on a neighborhood of x,t. Therefore (6a) holds simultaneously (same constant M for $O\left((\Delta_i t)^{3/2}\right)$) for all the $\Delta_i E$. This is needed so we can write $O\left((\Delta t)^{3/2}\right)/\sqrt{n}$ in (6b).

APPLICATION

The derivation shows the approximate amount of risk from the hedge if the adjustment is not continuous. Define R, the (random variable) rate of return during Δt (no interim adjustments) by

$$\Delta E = E\ R\ \Delta t\ . \tag{8}$$

Then from (5), letting $\mathcal{E}(X)$ be the expectation of a random variable X, it follows that

$$\begin{aligned} E(R-r)\Delta t &= E\ R\ \Delta t - E\ r\ \Delta t = \Delta E - \mathcal{E}(\Delta E) \\ &= \left(\tfrac{1}{2}w_{11}v^2x^2[Z^2-1]\right)\Delta t/w_1 + O\left((\Delta t)^{3/2}\right) \end{aligned} \tag{9}$$

whence

$$R - r = \frac{-w_{11}v^2x^2[Z^2-1]}{2(w_1x-w)} + O\left((\Delta t)^{1/2}\right) . \qquad (10)$$

Thus the <u>rate</u> of return R between adjustments is risky and the distribution of R about the mean r depends mainly on the coefficient $K = -w_{11}v^2x^2/2(w_1x-w)$ and, for small Δt, not much on Δt. However, the R's between successive adjustments are independent so, as the Δt steps between adjustments tends to zero, the risk tends to zero too, by the law of large numbers.

To study R, rewrite K in (10):

$$K = \frac{-w_{11}v^2x^2}{2(w_1x-w)} = \frac{w_2 - rw + rxw_1}{w_1x - w} = \frac{w_2}{w_1x - w} + r .$$

Note that $w_1x - w = N(d_1)x - w = ce^{r(t - t^*)}N(d_2)$ and

$w_2 = - ce^{r(t - t^*)}\left\{rN(d_2) + vN'(d_2)/2\sqrt{t^* - t}\right\}$. This gives

$$K = \frac{vN'(d_2)}{2\sqrt{t^* - t}\ N(d_2)} . \qquad (11)$$

Thus

$$R - r = K[Z^2 - 1] + O\left((\Delta t)^{1/2}\right)$$

where K is as in (11) and $Y = Z^2 - 1$ has the density

$$f_Y(y) = \frac{1}{\sqrt{y+1}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(y+1)}{2}\right\},\ y > -1$$

and 0 otherwise.

Note that, neglecting $O\left((\Delta t)^{1/2}\right)$, R satisfies $\mathcal{E}(R) = r$, $\sigma^2(R) = K^2\, \mathcal{E}\left((Z^2 - 1)^2\right) = 2K^2$, and (since $K < 0$) R is in the interval $-\infty < R < r - K$. A study of K will indicate the risk if the hedge is not adjusted during time Δt.

Table 3 is helpful in studying K. It suggests that $N'(d_2)/N(d_2) \sim -d_2$ as $d_2 \to -\infty$. This is correct and can be proved using asymptotic series for $N(x)$. For insight into the behavior of K, consider the special case $r - v^2/2 = 0$. This would be true, for instance, if $r = .07$ and $v = \sqrt{.14} = .37 \cdots$. Then setting $s^2 = v^2(t^* - t)$, $d_2 = \ln(x/c)/s = a$ where $\ln(x/c) = as$. Then $K \sim v^2 a/2s$ as $a \to -\infty$ or $K \sim 0.07a/s$.

After this appendix was written, Black sent me Black and Scholes [3,4], which I had not previously seen. These preliminary versions of Black and Scholes [6] present a Taylor series approach to the derivation of their model. It is close in spirit to this appendix.

TABLE 3

$N'(d_2)/N(d_2)$ as a Function of d_2.

d_2	$N'(d_2)$	$N(d_2)$	$N'(d_2)/N(d_2)$
-6	6.0800E-09	9.9000E-10	6.14
-5	1.4867E-06	2.8670E-07	5.19
-4	1.3383E-04	3.1671E-05	4.23
-3	4.4318E-03	1.3499E-03	3.28
-2	5.3991E-02	2.2750E-02	2.37
-1	2.4197E-01	1.5866E-01	1.53
0	3.9894E-01	5.0000E-01	0.80
1	2.4197E-01	8.4134E-01	0.29
2	5.3991E-02	9.7725E-01	0.06
3	4.4318E-03	9.9865E-01	0.00
4	1.3383E-04	9.9997E-01	0.00
5	1.4867E-06	1.0000E+00	0.00
6	6.0800E-09	1.0000E+00	0.00

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