Mathematics of Gambling

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Blackjack With a Tenless Deck

What is the player's expectation in blackjack against a tenless deck? What does he do when he uses (a) the optimal zero memory strategy; or (b) the optimal full-deck strategy (the "basic strategy"), which is definitely not optimal against a tenless deck?

In our September 1980 "Laying it on the Line," I said the best strategy for the tenless deck gains 1.62%. Julian Braun prompted me to check my records, and I found the facts to be somewhat different.

In 1960 I calculated the (approximate) best strategies and player edge for a variety of decks. This included several decks in which only the number of tens were varied. These were used to produce the ten-count method which I casino-tested in April, 1961. (That appeared in the 1962 edition of *Beat the Dealer*.)

In 1962, I made further extensive computer runs for various deck variations, and I computed the strategy and edge for the tenless deck. The result for the player's edge was +1.543%. The approximations caused the computed player edge to be a little smaller than it would be for an exact program. When Julian Braun further refined my program, as described in his paper, *Dr. Thorp's Arbitrary Subsets Program*, he got the more exact figure of 1.62%.

My tables for the tenless deck correspond to those in the Appendix of *Beat the Dealer* for the complete deck (basic strategy). From my tenless deck tables, I could calculate by hand, after several hours work, the loss in player expectation if I use full deck basic strategy to double down and to split pairs. The drawback with these tables is that I can't find the loss due to the wrong standing numbers.

Peter Griffin gives the exact answer for the infinite deck. He has supplied the first quantitative evidence that I know of, and it strongly supports his estimate of about −30%. This is for the player using basic strategy versus a single tenless deck. Griffin writes:

> *First I'll give you a quick fix, but not the answer to your specific question about single deck. I presume you wanted a calculation for both optimized strategy and basic strategy. My infinite deck blackjack program which searches for optimal strategy produced an advantage of 2.9% with repeated pair splitting. When I used* basic strategy *for the same tenless deck (no double after split) I got −28.8%. Now, this all is exact and takes virtually no time.*

Griffin's tenless deck strategy yields about −30%.

> *I have the capacity to run off the 36 card tenless deck and produce the absolutely exact player expectation from fully optimized strategy (no preprogramming, suggesting any restraints on how the hand should be played), but this would take quite a bit of time (the tens being out of the deck saves hardly any computer time), particularly for pair splits where there is much bizarre activity. (In infinite deck one splits fours against an ace!). Just to give you a feeling for the matter, I left out pair splitting and fully optimized against dealer 4 and dealer 8. Player's expectations were −1.4% and +5.9% respectively. Then I went back and made the computer stand on 12 and above and double all the normal hands against the 4; I had it double 10 and 11 against 8 and also stand on 17. Results were −47.2% and −5.0% respectively.*
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> *All this leads me to the following estimates for single deck: conventional basic strategy against a tenless deck would give an expectation very close to −30%, while fully optimized strategy would give something in excess of +2%. Note this is a little higher than you have (1.6%), but I've found higher expectations than yours, Braun's etc. for single deck, and half deck play, when* fully optimized. *I've found a minisculely lower expectation than the Manson NC State people quoted for 4 deck blackjack, and trace it to mistakes they made failing to correct drawing probabilities when dealer had an ace or ten up. I'm fairly confident in my estimates at the beginning of this paragraph, but the pair splitting time seems just too prohibitive. Note result is consistent with infinite deck findings.*
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> *A last note on the tenless single deck. I finished splitting the pairs and got a figure of +1.86% when exact composition dependent strategy is applied. What conventional basic strategy would yield will re-*

TABLE A

Q(10) One Deck	f	Thorp 1959-1962	Braun 1966	Insurance Adds	Thorp + Insurance	Braun + Insurance
0	0.000	1.543	1.62	0	same	same
1	.027	0.182		0	same	same
2	.053	− .792		0	same	same
3	.077	−1.533		0	same	same
4	.100	−2.137	−2.14*	0	same	same
5	.122	−2.643	−2.64*	0	same	same
6	.143	−3.033	−2.99	0	same	same
7	.163	−3.235		0	same	same
8	.182	−3.251	−3.13	0	same	same
9	.200	−3.070		0	same	same
10	.217	−2.787	−2.66*	0	same	same
11	.234	−2.451		0	same	same
12	.250	−2.059	−1.85	0	same	same
13	.265	−1.627		0	same	same
14	.280	−1.153		0	same	same
15	.294	−0.688		0	same	same
16	.308	−0.211	0.13	0	same	same
17	.321	0.257		0	same	same
18	.333	.732		0.094	.826	
19	.345	1.173				
20	.357	1.570	1.89	0.325		2.22
21	.368	2.005				
22	.379	2.422		0.544	2.966	
23	.390	2.820				
24	.400	3.201	3.51	0.734	3.935	4.24
25	.410	3.619				
26	.419	4.021		.899	4.920	
27	.429	4.427				
28	.438	4.805	5.06*	1.042	5.847	6.10
29	.446	5.191				
30	.455	5.553	5.82*	1.166	6.719	6.99
31	.463	5.898				
32	.471	6.225	6.48*	1.273		
33	.478	6.546				
34	.486	6.861		1.366		
35	.493	7.158				
36	.500	7.438	7.66	1.448	8.886	9.11
54	.600			1.823		
70	.660			1.887		
71	.663			1.887		
84	.700			1.863		
144	.800			1.570		
324	.900			0.949		
684	.950			0.515		
3564	.990			0.109		
ALL	1.000	0.000	0.000	0.000	0.000	0.000

*These values may not be exact.

main a matter of conjecture, but I'll stick with −30%, at least in the leading digit.

Note: There are several levels of "basic" strategy. At the crudest level, the player uses the same hitting and standing strategy for all hands with the same hard total or the same soft total. Also, his strategy does not change as he draws cards even though it is sometimes better to do so.

A second improved level considers the precise cards which the player was initially dealt. At this level, the one deck basic strategy, (with "typical" blackjack rules as in *Beat the Dealer*), would stand on 7, 7 versus a dealer 10 up card, but draw for many two card totals of 15 and 16.

By "fully optimized" I believe Peter is referring to a third (and highest) basic strategy level. Here, the player uses all the information about the cards he has drawn in his hand. His strategy will in some cases change as the hand unfolds. This level is simply optimal card counting play on the first hand dealt.

A version of basic strategy, which uses more information, has been named to a higher level. The advantage to the player is greater, because of the additional information he uses.

I had guessed that full deck basic strategy would most likely cut the +1.6% player edge by less than 1.6%. Sklansky guessed that it would be cut much more than 1.6%, and perhaps by the amount one would expect from "linearity."

I interpret "linearity" to mean that the number of tens alone is varied, and full deck basic strategy is always used; then the change in the player advantage is proportional to the change in the fraction of tens. (For a highly theoretical discussion of linearity in blackjack, see Peter Griffin's book, *The Theory of Blackjack.*)

In one full deck, the number of tens Q(10) is 16 and the fraction of tens is 16/52 = 4/13 = 0.3077. The player advantage is +0.13%. With Q(10) = 12, the advantage with the somewhat different best strategy is −1.85% (*Beat the Dealer*, revised, page 48). The fraction of tens is 12/48 = 1/4 = 0.2500. Thus, a decrease in the fraction of tens of 0.3077 − 0.2500 = 0.0577 gives a reduction in advantage of 0.13% + 1.85% = 1.98%.

If one deck basic strategy were used instead, the reduction would be a little larger. Under linearity, we would expect Q(10) = 0. That decreases the fraction of tens by 0.3077 from that in a full deck causing a decrease in basic strategy advantage. This advantage would be a little larger than (0.3077/0.0577) x 1.98% = 10.56%, or an expectation a little worse than 0.13 − 10.56% = −10.43%.

For those who worry about how much worse the true linearity estimate is than the −10.43% upper bound, we can use the Q(10) = 20 data to get a *lower* bound. For Q(10) = 20 we have a fraction of tens 20/56 = 5/14 = 0.3571 for an increase of 0.0495.

The player advantage changes from 0.13% to 1.89% for an increase of 1.76%. It will be a little smaller with basic strategy. A decrease in the tens fraction from 0.3077 (full deck) to 0 ought to give a decrease of a little less than (0.3077/0.0495) x 1.76% = 10.94%. Thus, the linearity estimate for a tenless deck is a player expectation that is a little higher than 0.13% − 10.94% = 10.81%.

For very large changes in the fraction of tens, linearity may give a very poor estimate of player expectation in full deck basic strategy usage. Further, the table in *Beat the Dealer*, revised, page 48 shows linearity way off for the optimal strategy and large changes in the fraction of tens.

Note that as the fraction of tens increases, the player advantage increases, up to a fraction 36/72 = 0.500 of tens. Linearity would imply this increase continues, but in the extreme case where the deck is all tens (the fraction is 1.00), the player advantage is exactly 0.00%. That is, when all hands total 20 regardless of strategy, excluding suicidal double downs. Therefore, the curve of player advantage versus fraction of tens must eventually peak and turn down again as the fraction of tens increases from 0.500 to 1.00.

There are several levels of "basic" strategy. At the crudest level, the player uses the same hitting and standing strategy for all hands with the same hard total or the same soft total.

As we continue to add tens to a single deck, when does the player advantage hit a maximum (a) without insurance and (b) with insurance? What is the value of this maximum advantage? As far as I know, no one has yet answered these questions, though it can be done readily with existing computer programs such as Thorp Braun or Griffin.

Table A presents the data I have. The behavior between f = 0.5 and f = 1.0 is not known, except that the curve should increase to a maximum for f above 0.5. As f increases further to 1.00 it should decline to 0.

If T is the number of tens (called Q(10) in Table A) in the (modified) single deck, then for T of 20 or more, the advantage which insurance adds is $2(2T - 35)/(T+36)(T + 35)$. Insurance also adds an advantage for T = 18 and T = 19. As T increases, the extra advantage from insurance increases too, until at T = 70 it reaches its maximum value of 1.887%. Table A shows the same value at T = 71. However, the T = 71 value is less in the ninth decimal place. **9t**