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The Mathematics of Gambling

Backgammon Tournament Play: Game Theory Solves Experts' Puzzle

by Edward O. Thorp

Backgammon continually puzzles the players with new situations to judge. This often leads to debate, even among experts. It is generally not realized that many backgammon problems, even those that puzzle experts, can be answered exactly through mathematics. I give an example this month, taken from the new backgammon magazine, *Gammon.*

Here is the problem *Gammon* put to several experts:

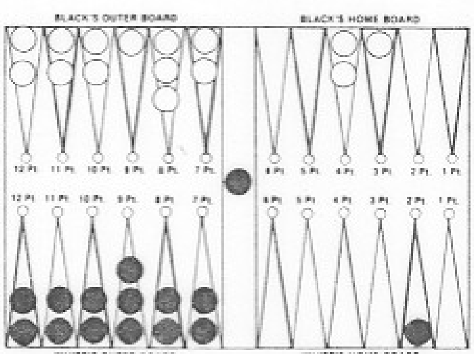
Before we discuss the problem itself, you need to know that because of the scoring, the strategy in match or tournament play is often different from that in money play. The situation is like that in bridge, where different plays may be correct in the same deal, depending on whether rubber bridge scoring, duplicate bridge scoring, or international match point scoring (IMP) is used. Here is the simplest and best known example. Suppose you are behind 13 to 14 in 15-point match. Then you should double at your first opportunity, even if you have the poorer game. Why? If you don't double, you win the match either by gammoning in this game or by winning both one point in this game and winning the next game. If you do double and your opponent accepts, then you only have to win this game to win the match. So you are better off. If you do double and he drops instead, then the score becomes 14 to 14, and you only have to win the next game to win that match. You again are better off.

Suppose you neglected to double early in the game with the score 13-14, and you now find that you have a "good" chance to gammon your opponent. Should you double? Maybe not. Let p_1 be the probability you will win a single game, and let p_2 be the probability that you will gammon or backgammon your opponent in this game. Let w be the probability that you will win the next game. (For two players of equal skill, we might expect $w = ½$.) If you do not double, then the probabilities are p_2 that you win the match in this game, p_1 that the score is 14-14 after this game, and $1 - p_1 - p_2$ that you lose this game and the match. The total probability that you win the match is $p_2 + p_1 w$. If you do double and your opponent folds, the score becomes 14-14, and the probability is w that

Even the Experts Disagree

This situation occurred during the open division of the Las Vegas Amateurs. Arthur Dickman was playing Sandy Carlston. Both are top professionals. At this position, Dickman (black), who was behind 13-12 in the 15-point match, considered doubling. Would you double if you were black?

Here's What the Experts Said



Billy Eisenberg: Yes, but only because of the score. It's tough, but black's chance to win the match will never be better. Black has gammon possibilities if he rolls a 4. If he doesn't double and doesn't hit, he's 12-14 in a 15-point match and only about 2-5 to win. And, of course, white takes.

Chuck Papazian: No! 12-14 isn't so bad. Only a 2½-1 underdog. And if black hits but doesn't get the gammon, black is still only 14-13. It's insanity to double.

Barclay Cooke: No. I wouldn't want to put the match on a 25-11 shot against me. Even at 12-14, it's not over.

Roger Low: It depends on whether or not black is going to roll a 4.

P.S.: Dickman didn't double, but he didn't roll a 4 either. And Carlston eventually won the match.

you win the match. If you double and your opponent accepts, this game decides the match, and the probability you win is $p_1 + p_2$.

If you double, your opponent will act to minimize the chance you win the match. Therefore, he accepts if $p_1 + p_2 < w$, folds if $p_1 + p_2 > w$, and it doesn't matter if $p_1 + p_2 = w$. Therefore, if you double, your chance to win the match is the lesser of $p_1 + p_2$ and w, or $min(p_1 + p_2, w)$. You will double if this is better than $p_2 + p_1w$, you will not double if this is worse than $p_2 + p_1w$, and it doesn't matter when $min(p_1 + p_2, w) = p_2 + p_1w$.

Table 1 illustrates this point. The asterisks mark the best plays by both players. The last column gives the chance you will win, assuming this best play. You can see from the various examples or from the formulas that you should generally double if you're behind or even, but if you're ahead, you may not want to double, especially if you are far ahead, have good gammon prospects, or are a much weaker player than your opponent. If you have good estimates for p_1, p_2, *and* w, then you can compute what to do.

The dramatic difference between this situation and money play is that here you should always double when you're behind; whereas, in money play, you should rarely double when you're behind.

Note: An exception, pointed out by Bill Granoff, is the "Kander paradox," which may arise when the Jacoby rule is in effect. The Jacoby rule is an option which is sometimes used to speed up match play. It prohibits either player from winning more than one point in a game, unless the cube has been turned. Here is a simplified example to indicate how the Kander paradox works. Suppose the Jacoby rule is being used, the cube has not been turned, and that it is black's turn to roll. Suppose the probabili-

Situation	P_1	P_2	W	(a) If You Don't Double P_2+P_1W	If You Double And Opponent: (b) Folds W	(c) Accepts P_1+P_2	Your Chance To Win Match With Best Strategy	Strategy: Do You Double?	Does He Accept?
Start of Game	0.4	0.1	0.5	0.30	0.5**	0.5**	0.5	Yes	Either
You're Losing	0.2	0.05	0.5	0.15	0.5	0.25**	0.25	Yes	Yes
You're Ahead	0.6	0.2	0.5	0.50*	0.5**	0.80	0.50	Either	No
You're Somewhat Ahead	0.5	0.2	0.5	0.45	0.5**	0.70	0.50	Yes	No
You're Far Ahead	0.6	0.3	0.5	0.60*	0.5**	0.90	0.60	No	No
You're Ahead, he's Better	0.6	0.2	0.4	0.44*	0.4**	0.80	0.44	No	No
You're Somewhat Ahead, He's Better	0.5	0.2	0.4	0.40*	0.4**	0.7	0.40	Either	No
You're Somewhat Ahead, He's Much Better	0.5	0.2	0.3	0.35*	0.3**	0.7	0.35	No	No

Table 1

When to double if you trail 13-14 in a 15-point match. In the current game, p_1 is your chance to win one point, and p_2 is the chance you will gammon or bankgammon your opponent. Your chance to win a subsequent game is w. The table shows your chance to win the match (a) if you don't double, (b) if you double and opponent folds, or (c) if you double and opponent accepts. Opponent picks (b) or (c), according to which gives you the smallest chance. A ** marks his best choice. If this gives you a smaller chance to win than if you don't double, then you pick (a), and this is indicated by a *. The last column gives your corresponding chance to win the match with these (best) choices by you and your opponent.

CASE	PROBABILITY THAT ● WINS	PROBABILITY THAT ○ WINS	STRATEGY
A ● Does Not Double	$1/4 + P_1/4 + P_2/2 - g\ P_2/2$	$3/4 - P_1/4 - P_2/2 + g\ P_2/2$	● Should not double because $p < 1/2$
B ● Doubles, ○ Accepts	$3P_1/4 + P_2 - g\ P_1/2$	$1 - 3P_1/4 - P_2 + g\ P_1/2$	○ Should accept a double if it is offered.
C ● Doubles, ○ Folds	1/2	1/2	

Table 2

Results of tree diagram analysis of experts' problem, where the probability black wins a single game is p_1, a gammon is p_2, and a backgammon is *0*. The chance of winning a single point in any subsequent game is $\frac{1}{2} - g$ *for each player and* g that each player gets more than a single point.

ty black will win a simple game is .05, a gammon is .40, and a backgammon is 0. Then black is the underdog, with a .45 probability of winning one point and 0.55 or losing one point for an expected gain of −0.10 points. Suppose that the probability that white wins a single game is 0.55, and there is no chance for him to win a gammon or backgammon (e.g., black has already taken a man off). If black doubles, his expected gain is $2\times(1 \times .05 + 2 \times .40 - 1 \times .55) = 0.60$. Therefore, that act of doubling converts black from underdog to favorite. I understand that Kander discovered and correctly applied this idea in the course of a game.

We now return to the experts' problem. Make these assumptions:

- Let p be the (unknown) probability that black wins this specific game, with p_1 the probability it is a single game, p_2 the probability that it is a double game (gammon), and assume that p_3, the probability of a triple game (backgammon), is negligible. Thus, $p=p_1+p_2$ and $p_3=0$.
- Suppose the probability that either black or white wins any giv-

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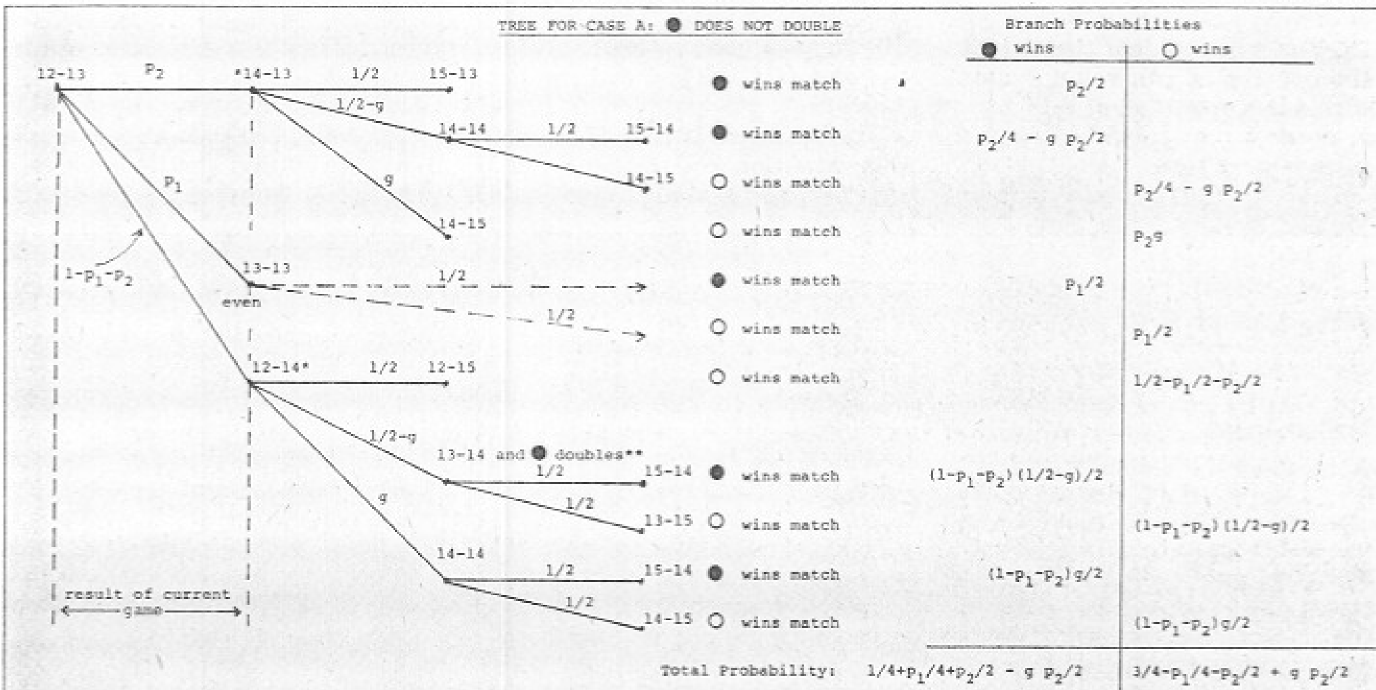
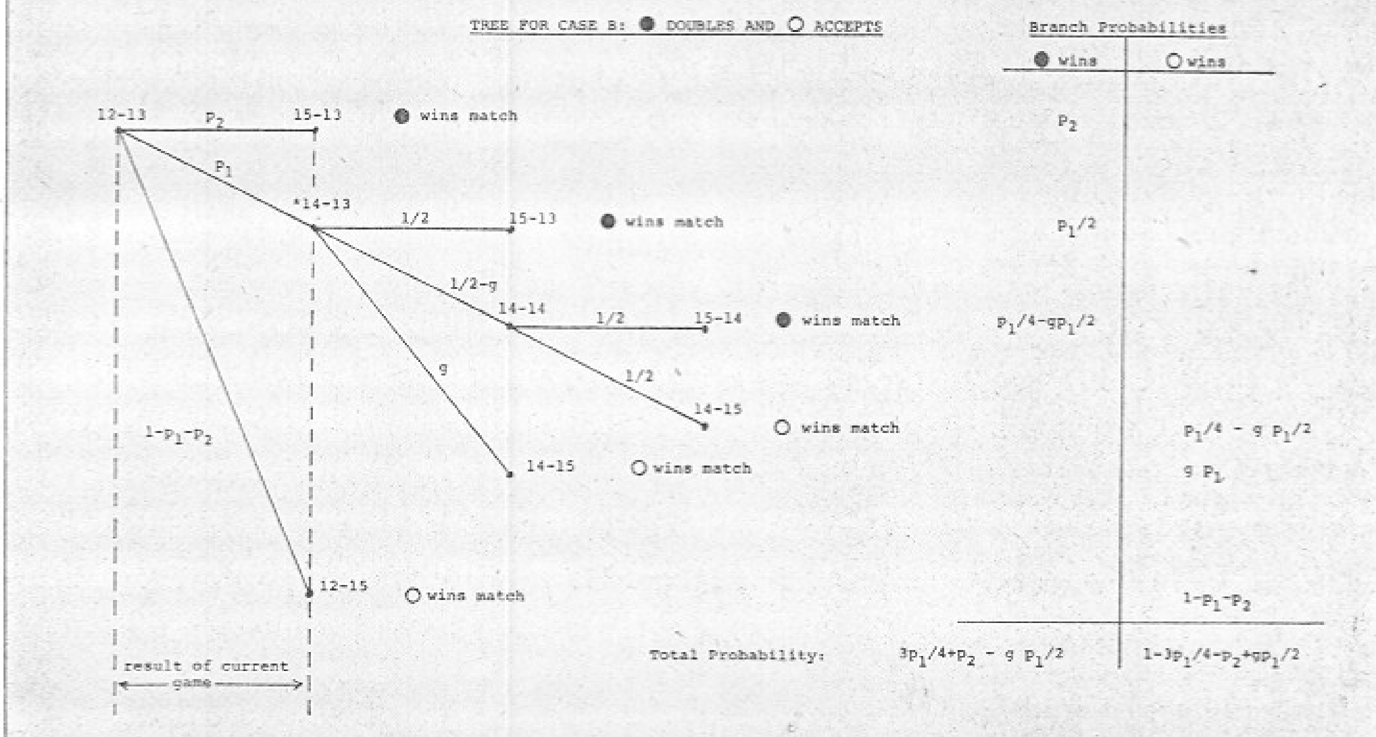


*Crawford rule prevents opponent from doubling in next game. Winning scores listed as 15 may actually be more if the winner scores a gammon or backgammon.

**At 13-14, white has a slight advantage from his free pass, so he has slightly more than a 50 percent chance to win (see text). The effect is small, and we have neglected it.



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en subsequent game is ½. This is not unreasonable in light of the match score and the acknowledged skill of the players. Suppose the probability that black or white gets a gammon or backgammon in any subsequent game is g. I am told that a reasonable guess for g is *0.07*, so about 14 percent of the games might be expected to end in gammon or backgammon. Then $\frac{1}{2} - g$ is the probability that either black or white wins only one point in a subsequent game.

We now calculate the match chances for black for the three cases:

A. Black does not double.
B. Black doubles and white accepts.
C. Black doubles and white folds.

Case C is easy. The score becomes 13-13, and the players have equal skill, so they each have a 50 percent chance to win the match.

Case A is illustrated by a "tree diagram." This is a standard tool in game theory, and it got its name because of the way it looks. Here is how it works. Start at the upper left "dot" or "vertex," which is labelled 12-13, the current match score. The upper branch going to the right is labelled p_2 and represents the probability that black gammons white in the current game. The next vertex is labelled *14-13, the score after the gammon. The asterisk on the 14 means that the Crawford rule is in effect: "Crawford rule: A rule invented by the late John Crawford, used in tournaments, under which a player is prevented from offering a double in the first game after his opponent reaches a score one short of the number of points to which the match is being played." (From *Phillip Martyn on Backgammon*, page 186.) Therefore, white cannot double on the next game. The upper branch continues with the outcome 15-13, which corresponds to the next game, and thus the match, being won by black. The branch is labelled ½ because we have assumed that this is the probability black wins the next game. Now multiply the two probabilities p_2 and ½ to get $p_2/2$, the probability that black wins the match by this series of events. The rest of the diagram in interpreted similarly.

The rightmost two columns give the probabilities that black or white wins the match via the corresponding series of branches. Adding up these probabilities, we find that the probability that black wins, if he doesn't double, is $\frac{1}{4} + p_1/4 + p_2/2 - g\,p_2/2$ and the probability that white wins in this case is $3/4 - p_1/4 - p_2/2 + g\,p_2/2$.

Note for case A: At the point of the tree where it is 13-14 and black doubles, we have given the chances of winning the match as ½ each for black and white. Actually, as we noted there, white has a slight advantage and here's why. When black doubles at his first opportunity, the game will, in general, already be slightly unequal. For instance, white may go first and roll something good like 3-1 or something poor like 2-1. Or black may go first, and by the time he can double on his second roll, the game may have become noticeably unequal. To illustrate, suppose that half the time black has a 48 percent win probability by the time he can double and white accepts, and the other half of the time he has a 52 percent win probability and white folds, whence the score goes to 14-14 with equal chances for each to win the match. Then black has probability *48%/2 + 50%/2=49%* to win the match. That's how the "free pass" gives white a little edge.

The "Holland rule" has been used in some tournaments to offset this. It specifies that in the game following the "Crawford game," the player who is behind in the match must wait three turns before he can double. Since the game is likely to have become even more unequal by then, this increases white's edge. In the experts' problem, the Crawford rule was in effect, whereas the Jacoby and Holland rules were not.

To see whether black should double, we now have to examine case B and see whether it gives him a better chance to win. Here we find that if black doubles and white accepts, then black wins with probability $3p_1/4 + p_2 - g\,p_1/2$, and white wins with probability $1 - 3p_1/4 - p_2 + g\,p_1/2$.

Table 2 summarizes our results so far. We see that white should accept a double if doing so decreases the black win probability, that is if $\frac{1}{2} > 3p_1/4 + p_2 - g\,p_1/2$. Now the total win probability for black in the current game is $p = p_1 + p_2$, and it seems clear that $p < \frac{1}{2}$, i.e., that black is the underdog in the current game. But $\frac{1}{2} > p_1 + p_2$ means $\frac{1}{2} > 3p_1/4 + p_2 - g\,p_1/2$, since $3p_1/4 + p_2 - g\,p_1/2$ is smaller than $p_1 + p_2$. Therefore, white should accept a double if it is offered.

Next question: Should black offer a double? He should if it increases his win probability, that is if $3p_1/4 + p_2 - g\,p_1/2 > \frac{1}{4} + p_1/4 + p_2/2 - g\,p_2/2$. This is equivalent to $(1 - g)(p_1/2 + p_2/2) > \frac{1}{4}$ or $(1 - g)(p_1 + p_2) > \frac{1}{2}$ or $(1 - g)\,p > \frac{1}{2}$. But $p < \frac{1}{2}$, so black should not double.

If black does double, how much does he lose? His probability of winning the match drops by $(\frac{1}{4} + p_1/4 + p_2/2 - g\,p_2/2) - (3p_1/4 + p_2 - g\,p_1/2) = \frac{1}{4} - p_1/2 - p_2/2 + g(p_1 - p_2)/2 = \frac{1}{2}(\frac{1}{2} - p) + g(p_1 - p_2)/2$. Therefore, if, for example, the probability of black winning the current game were $\frac{1}{3}$, then black's probability to win the match decreases by more than $1/12$. Since it was pretty small to begin with, this is a major reduction in black's chances, and **doubling is a gross error.**

There are many amazing things about this analysis. First, the analysis itself is mathematically simple (you may not agree if you are not skilled in this area). Second, the result is a surprise. It shows that the answer to the experts' puzzle isn't even close. Furthermore, the probabilities of gammon or backgammon in this or subsequent games of the match have little effect on the answer. All we need to know to get the answer is that black is the underdog in the current game and that the two players are of (approximately) equal skill.

In the problem, Chuck Papazian said that black would be a $2\frac{1}{2}$:1 underdog if the match score became 12-14. This is not correct. Using a tree diagram, I find that black's chance to win the match is $(\frac{1}{2} - p_2 - p_3)(\frac{1}{2} - d) + \frac{1}{2}p_2 + p_3$.

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PROGRESSIVE SYSTEMS

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sion, the bettor will be ahead. So I can afford to increase my bets and take my chance that the streak will continue. If I win, I win big. And, if I lose, well, I haven't really lost anything, because I was "playing with the house's money."

Comment: One silliness leads to another. You are never playing "with the house's money," unless you stole it or are one of the casino owners. Every time you make a bet, you are risking your own money.

The odds of any game are calculated by totalling the number of times you can expect to win and offsetting them with the number of times you can expect to lose. The excess of the loss, over a period of time, constitutes the house percentage. In craps, once again, the advantage is 1.4 percent. Now, if you don't think of your wins as committed to offsetting your losses, but instead think of them as profit—a.k.a. house money—which you can bet as wildly as you want without risk, the house percentage will no longer be 1.4 percent. This figure was achieved by balancing wins and losses. Remove the wins, and the house percentage becomes astronomical.

You can't have it both ways. Your wins are calculated as part of the whole game and are part of what keeps the house percentage low. Or else they are considered separately as profit, in which case the house take on all the rest of your bets (nothing but losses and "pushes") simply goes off the scale.

As the chart shows, the number of times you are "betting with house money" balances the number of times you aren't. In the long run, all you accomplish by using a progressive system is to increase the amount wagered, thus increasing the total the house collects as its "vigorish."

The casinos know this very well. I often wonder whether they don't deliberately contribute articles and books which promote progressive systems, precisely because they know that they will make more off the players who use them. A pit-boss once told me that the players they like the least are those who grind it out with unit bets. "We are never going to get rich off those players," he said.

As a typical player once said to me, "I didn't come all the way here to Las Vegas to sit hour after hour making unit bets, even if that is the best way. I came out here for action!"

Temperament is important. There are those who would rather have an occasional giant win, which they will remember all their lives, even though they may realize that such wins are more than nibbled away in between times. The excitement of the big parlay when everything is breaking right, the sense of near omnipotence, and the feeling that one is really a member of a special breed is what makes it all worthwhile. This is what they work for, and wait for, and dream about. Otherwise, they wouldn't come. Nobody should begrudge them their own particular brand of pleasure. After all, the name of the game is enjoy. May the Lords of Chance bless them. We know the casino certainly does.

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where the chance that black gets a gammon is p_2 and a backgammon is p_3. The advantage to white at 13-14 due to the free pass is d. As an example, take $d=.02$, $p_2=.06$ and $p_3=.02$. Then black's chance to win the match is 0.2516, and the corresponding odds are 2.9746:1. I think that plausible choices for d, p_2, and p_3 will generally give odds much closer to 3:1 than to 2½:1.

However, with no Crawford rule, a tree analysis gives black's chance to win the match as $\frac{1}{4} + (p_2 + p_3)/2$. If $p_2 + p_3 = 1/14 \doteq .07$, then this is exactly 2½:1. If $p_2 + p_3 = .08$, black's chance is about 2.45:1. Thus, when Papazian answered the question, he may have believed that the Crawford rule was not in effect. Don Smolen independently pointed this out.

The general theory of doubling in match play, of which this column is an example, is complex. I hope to present it in later columns. I thank Bill Granoff for suggesting several improvements in this article.