The Mathematics of Gambling

Backgammon Tournament Play: Game Theory Solves Experts’ Puzzle

by Edward O. Thorp

Backgammon continually puzzles the players with new situations to judge. This often leads to debate, even among experts. It is generally not realized that many backgammon problems, even those that puzzle experts, can be answered exactly through mathematics. I give an example this month, taken from the new backgammon magazine, Gammon.

Here is the problem Gammon put to several experts:

Even the Experts Disagree

This situation occurred during the open division of the Las Vegas Amateurs. Arthur Dickman was playing Sandy Carlston. Both are top professionals. At this position, Dickman (black), who was behind 13-12 in the 15-point match, considered doubling. Would you double if you were black?

Here’s What the Experts Said

Billy Eisenberg: Yes, but only because of the score. It’s tough, but black’s chance to win the match will never be better. Black has gammon possibilities if he rolls a 4, if he doesn’t double and doesn’t hit, he’s 12-14 in a 15-point match and only about 2-5 to win. And, of course, white takes.

Chuck Papazian: No! 12-14 isn’t so bad. Only a 2 ¼-1 underdog. And if black hits but doesn’t get the gammon, black is still only 14-13. It’s insanity to double.

Barclay Cooke: No. I wouldn’t want to put the match on a 25-11 shot against me. Even at 12-14, it’s not over.

Roger Low: It depends on whether or not black is going to roll a 4.

P.S.: Dickman didn’t double, but he didn’t roll a 4 either. And Carlston eventually won the match.

Before we discuss the problem itself, you need to know that because of the scoring, the strategy in match or tournament play is often different from that in money play. The situation is like that in bridge, where different plays may be correct in the same deal, depending on whether rubber bridge scoring, duplicate bridge scoring, or international match point scoring (IMP) is used. Here is the simplest and best known example. Suppose you are behind 13 to 14 in 15-point match. Then you should double at your first opportunity, even if you have the poorer game. Why? If you don’t double, you win the match either by gammoning in this game or by winning both one point in this game and winning the next game. If you do double and your opponent accepts, then you only have to win this game to win the match. So you are better off. If you do double and he drops instead, then the score becomes 14 to 14, and you only have to win the next game to win that match. You again are better off.

Suppose you neglected to double early in the game with the score 13-14, and you now find that you have a “good” chance to gammon your opponent. Should you double? Maybe not. Let \( p_1 \) be the probability you will win a single game, and let \( p_2 \) be the probability that you will gammon or backgammon your opponent in this game. Let \( w \) be the probability that you will win the next game. (For two players of equal skill, we might expect \( w = \frac{1}{2} \).) If you do not double, then the probabilities are \( p_2 \) that you win the match in this game, \( p \), that the score is 14-14 after this game, and \( 1 - p_1 - p_2 \) that you lose this game and the match. The total probability that you win the match is \( p_2 + p + w \). If you do double and your opponent folds, the score becomes 14-14, and the probability is \( w \) that
you win the match. If you double and your opponent accepts, this game decides the match, and the probability you win is $p_1 + p_2$.

If you double, your opponent will act to minimize the chance you win the match. Therefore, he accepts if $p_1 + p_2 < w$, folds if $p_1 + p_2 > w$, and it doesn’t matter if $p_1 + p_2 = w$. Therefore, if you double, your chance to win the match is the lesser of $p_1 + p_2$ and $w$, or $\min(p_1 + p_2, w)$. You will double if this is better than $p_1 + p_2$, you will not double if this is worse than $p_1 + p_2$, and it doesn’t matter when $\min(p_1 + p_2, w) = p_1 + p_2$.

Table 1 illustrates this point.

The asterisks mark the best plays by both players. The last column gives the chance you will win, assuming this best play. You can see from the various examples or from the formulas that you should generally double if you’re behind or even, but if you’re ahead, you may not want to double, especially if you are far ahead, have good gammmon prospects, or are a much weaker player than your opponent.

If you have good estimates for $p_1$, $p_2$, and $w$, then you can compute what to do.

The dramatic difference between this situation and money play is that here you should always double when you’re behind; whereas, in money play, you should rarely double when you’re behind.

Note: An exception, pointed out by Bill Granoff, is the “Kander paradox,” which may arise when the Jacoby rule is in effect. The Jacoby rule is an option which is sometimes used to speed up match play. It prohibits either player from winning more than one point in a game, unless the cube has been turned. Here is a simplified example to indicate how the Kander paradox works. Suppose the Jacoby rule is being used, the cube has not been turned, and that it is black’s turn to roll. Suppose the probabili-

### Table 1

<table>
<thead>
<tr>
<th>Situation</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$w$</th>
<th>$p_1 + p_2$</th>
<th>$w$</th>
<th>$p_1 + p_2$</th>
<th>$p_1 + p_2$</th>
<th>$p_1 + p_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Start of Game</td>
<td>0.4</td>
<td>0.1</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>You’re Losing</td>
<td>0.2</td>
<td>0.05</td>
<td>0.5</td>
<td>0.35</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>You’re Ahead</td>
<td>0.6</td>
<td>0.2</td>
<td>0.5</td>
<td>0.55</td>
<td>0.5</td>
<td>0.55</td>
<td>0.55</td>
<td>0.55</td>
</tr>
<tr>
<td>You’re Somewhat Ahead</td>
<td>0.5</td>
<td>0.2</td>
<td>0.5</td>
<td>0.45</td>
<td>0.5</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>You’re Far Ahead</td>
<td>0.6</td>
<td>0.3</td>
<td>0.5</td>
<td>0.6</td>
<td>0.5</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>You’re Ahead, he’s better</td>
<td>0.6</td>
<td>0.2</td>
<td>0.4</td>
<td>0.55</td>
<td>0.6</td>
<td>0.55</td>
<td>0.6</td>
<td>0.55</td>
</tr>
<tr>
<td>You’re Somewhat Ahead, he’s Better</td>
<td>0.5</td>
<td>0.2</td>
<td>0.4</td>
<td>0.55</td>
<td>0.6</td>
<td>0.55</td>
<td>0.6</td>
<td>0.55</td>
</tr>
<tr>
<td>You're Much Better</td>
<td>0.5</td>
<td>0.2</td>
<td>0.3</td>
<td>0.55</td>
<td>0.6</td>
<td>0.55</td>
<td>0.6</td>
<td>0.55</td>
</tr>
</tbody>
</table>

### Table 2

When to double if you trail 13-14 in a 15-point match. In the current game, $p_2$ is your chance to win one point, and $p_3$ is the chance you will gammon or backgammon your opponent. Your chance to win a subsequent game is $w$. The table shows your chance to win the match (a) if you don't double, (b) if you double and opponent folds, or (c) if you double and opponent accepts. Opponent picks (b) or (c), according to which gives you the smallest chance. A ** marks his best choice. If this gives you a smaller chance to win than if you don't double, then you pick (a), and this is indicated by *. The last column gives your corresponding chance to win the match with these (best) choices by you and your opponent.

### Table 2

<table>
<thead>
<tr>
<th>CASE</th>
<th>WIN</th>
<th># WINS</th>
<th># WINS</th>
<th>STRATEGY</th>
</tr>
</thead>
<tbody>
<tr>
<td>A - Does Not Double</td>
<td>$\frac{1}{4} + \frac{p_1}{4} + \frac{p_2}{2} + g \frac{p_2}{2}$</td>
<td>$\frac{1}{4} + \frac{p_1}{4} + \frac{p_2}{2} + g \frac{p_2}{2}$</td>
<td>$\frac{1}{4} + \frac{p_1}{4} + \frac{p_2}{2} + g \frac{p_2}{2}$</td>
<td>Should not double because $p &lt; 1/3$</td>
</tr>
<tr>
<td>B - Doubles, Accepts</td>
<td>$3\frac{p_1}{4} + \frac{p_2}{2} - g \frac{p_1}{2}$</td>
<td>$1 - 3\frac{p_1}{4} + \frac{p_2}{2} - g \frac{p_2}{2}$</td>
<td>$1 - 3\frac{p_1}{4} + \frac{p_2}{2} - g \frac{p_2}{2}$</td>
<td>Should accept a double if it is offered.</td>
</tr>
<tr>
<td>C - Doubles, Folds</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$1/2$</td>
</tr>
</tbody>
</table>

**Table 2**

Results of tree diagram analysis of experts' problem, where the probability black wins a single game is $p_1$, a gammon is $p_2$, and a backgammon is $0$. The chance of winning a single point in any subsequent game is $\frac{1}{3} - g$ for each player and $g$ that each player gets more than a single point.
ty black will win a simple game is .05, a gammon is .40, and a backgammon is 0. Then black is the underdog, with a .45 probability of winning one point and 0.55 of losing one point for an expected gain of −0.10 points. Suppose that the probability that white wins a single game is 0.55, and there is no chance for him to win a gammon or backgammon (e.g., black has already taken a man off). If black doubles, his expected gain is $2 \times (1 \times .05 + 2 \times .40 - 1 \times .55) = .60$. Therefore, that act of doubling converts black from underdog to favorite. I understand that Kander discovered and correctly applied this idea in the course of a game.

We now return to the experts' problem. Make these assumptions:

- Let $p$ be the (unknown) probability that black wins this specific game, with $p$, the probability it is a single game, $p_d$ the probability that it is a double game (gammon), and assume that $p_d$, the probability of a triple game (backgammon), is negligible. Thus, $p = p_s + p_d$ and $p_d = 0$.

- Suppose the probability that either black or white wins any given

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**Crawford rule prevents opponent from doubling in next game. Winning scores listed as 15 may actually be more if the winner scores a gammon or backgammon.**

**At 13-14, white has a slight advantage from his free pass, so he has slightly more than 50 percent chance to win (see text). The effect is small, and we have neglected it.**
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en subsequent game is $1/2$. This is not unreasonable in light of the match score and the acknowledged skill of the players. Suppose the probability that black or white gets a gammon or backgammon in any subsequent game is $g$. I am told that a reasonable guess for $g$ is $0.07$, so about $14$ percent of the games might be expected to end in gammon or backgammon. Then $1/2 - g$ is the probability that either black or white wins only one point in a subsequent game.

We now calculate the match chances for black for the three cases:

A. Black does not double.
B. Black doubles and white accepts.
C. Black doubles and white folds.

Case C is easy. The score becomes 13-13, and the players have equal skill, so they each have a 50 percent chance to win the match.

Case A is illustrated by a "tree diagram." This is a standard tool in game theory, and it got its name because of the way it looks. Here is how it works. Start at the upper left "dot" or "vertex," which is labelled 12-13, the current match score. The upper branch going to the right is labelled $p_2$ and represents the probability that black gammons white in the current game. The next vertex is labelled $*14-13$, the score after the gammon. The asterisk on the 14 means that the Crawford rule is in effect: "Crawford rule: A rule invented by the late John Crawford, used in tournaments, under which a player is prevented from offering a double in the first game after his opponent reaches a score one short of the number of points to which the match is being played." (From Phillip Martyn on Backgammon, page 186.) Therefore, white cannot double on the next game. The upper branch continues with the outcome 15-13, which corresponds to the next game, and thus the match, being won by black. The branch is labelled $1/2$ because we have as-
cepts, then black wins with probability \(3p/4 + p_2 - g p_2/2\) and white wins with probability \(1 - 3p/4 - p_2 + g p_2/2\).

Table 2 summarizes our results so far. We see that white should accept a double if doing so decreases the black win probability, that is if 
\(p_2 > 3p/4 + p_2 - g p_2/2\). Now the total win probability for black in the current game is 
\(p = p_1 + p_2\) and it seems clear that \(p < 1/2\), i.e., that black is the underdog in the current game. But \(p_1 + p_2\) means 
\(3p/4 + p_2 - g p_2/2\), since \(3p/4 + p_2 - g p_2/2\) is smaller than \(p_1 + p_2\). Therefore, white should accept a double if it is offered.

Next question: Should black offer a double? He should if it increases his win probability, that is if 
\(3p/4 + p_2 - g p_2/2 > 1/2\). This is equivalent to 
\((1 - g)(p_2/2) > 1/4\) or 
\((1 - g)(p_1 + p_2) > 1/2\) or 
\((1 - g) p > 1/8\). But \(p < 1/2\), so black should not double.

If black does double, how much does he lose? His probability of winning the match drops by 
\((1 - p_1 + p_2 - p_2 - g p_2/2) = (3p/4 + p_2 - g p_2/2) - (1/4 + p_2 - p_2 + g p_2/2) = (1/4 + p_2 - p_2 + g p_2/2)\). Therefore, if, for example, the probability of black winning the current game were \(1/3\), then black’s probability to win the match decreases by more than \(1/12\). Since it was pretty small to begin with, this is a major reduction in black’s chances, and doubling is a gross error.

There are many amazing things about this analysis. First, the analysis itself is mathematically simple (you may not agree if you are not skilled in this area). Second, the result is a surprise. It shows that the answer to the experts’ puzzle isn’t even close. Furthermore, the probabilities of gammon or backgammon in this or subsequent games of the match have little effect on the answer. All we need to know to get the answer is that black is the underdog in the current game and that the two players are of (approximately) equal skill.

In the problem, Chuck Papazian said that black would be a 2:1 underdog if the match score became 12-14. This is not correct. Using a tree diagram, I find that black’s chance to win the match is 
\((1/2 - p_2 - p_2 + p_2)\).

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boss once told me that the players they like the least are those who grind it out with unit bets. "We are never going to get rich off those players," he said.

As a typical player once said to me, "I didn't come all the way here to Las Vegas to sit hour after hour making unit bets, even if that is the best way. I came out here for action!"

Temperament is important. There are those who would rather have an occasional giant win, which they will remember all their lives, even though they may realize that such wins are more than nibbled away in between times. The excitement of the big parlay when everything is breaking right, the sense of near omnipotence, and the feeling that one is really a member of a special breed is what makes it all worthwhile. This is what they work for, and wait for, and dream about. Otherwise, they wouldn't come. Nobody should begrudge them their own particular brand of pleasure. After all, the name of the game is enjoy. May the Lords of Chance bless them. We know the casino certainly does.

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where the chance that black gets a gammon is \( p_2 \), and a backgammon is \( p_3 \). The advantage to white at 13-14 due to the free pass is \( d \). As an example, take \( d = 0.02 \), \( p_2 = 0.06 \) and \( p_3 = 0.02 \). Then black's chance to win the match is 0.2516, and the corresponding odds are 2.9746:1. I think that plausible choices for \( d \), \( p_2 \), and \( p_3 \) will generally give odds much closer to 3:1 than to 2½:1.

However, with no Crawford rule, a tree analysis gives black's chance to win the match as \( \frac{1}{4} + (p_2 + p_3)/2 \). If \( p_2 + p_3 \leq 1/14 = 0.07 \), then this is exactly 2½:1. If \( p_2 + p_3 = 0.08 \), black's chance is about 2.45:1. Thus, when Papazian answered the question, he may have believed that the Crawford rule was not in effect. Don Smolen independently pointed this out.

The general theory of doubling in match play, of which this column is an example, is complex. I hope to present it in later columns. I thank Bill Granoff for suggesting several improvements in this article.