ATTEMPT AT A STRONGEST VECTOR TOPOLOGY

E. O. THORP, New Mexico State University

A set A in E is circled if $aA \subset A$ for all scalars a such that $|a| \leqq 1$. A set is radial at 0 if it contains a line segment through 0 in each direction. A vector topology for E is a topology such that addition and scalar multiplication are each jointly continuous. A local base is a fundamental system of neighborhoods of 0.

If E is an infinite dimensional real or complex linear space, the family of all circled sets radial at 0 is not a local base for a vector topology for E. This is observed in [1, I, Section 1, no. 9, ex. 6] and the example is repeated in [2, 5D], where the infinite dimensionality of E is essential to the argument.

We offer here the sharper observation that if E is a real or complex linear space, the family of all circled sets radial at 0 is not a local base for a vector topology for E iff $\dim E \geqq 2$.

First consider the case $E = E^2$, where E^2 is either real or complex two dimensional Euclidean space and $\{u_1, u_2\}$ is the usual orthonormal basis. For each x in E such that $|x| = 1$, let $f(x)$ be the magnitude $|(x, u_1)|$ of the inner product (x, u_1). Let A consist of a union of ray segments, one for each x such that $|x| = 1$. If $f(x)$ is 0 or irrational, let $[0, x] \subset A$. If $f(x) = p/q$ is a nonzero rational in lowest terms, let $[0, x/q] \subset A$. The fact that A is circled follows from $|(x, u_1)| = |(e^{i\theta}x, u_1)|$. Clearly A is radial at 0.

There is no set C radial at 0 with $C + C \subset A$. Suppose that there is such a C. Then there is an $\epsilon > 0$ such that $[0, \epsilon u_1]$ and $[0, \epsilon u_2]$ are in C. Therefore $C + C$ contains all ray segments $[0, x]$ where

$$x = t_1 u_1 + t_2 u_2, \qquad 0 \leqq t_1, t_2 \leqq \epsilon.$$

Letting

$$y = \frac{t_1 u_1 + t_2 u_2}{\sqrt{(t_1^2 + t_2^2)}}, \qquad \text{we have} \quad f(y) = \frac{t_1}{\sqrt{(t_1^2 + t_2^2)}}.$$

Let q be a prime such that $1/q < \epsilon$. If $t_1 = \epsilon/q$ and $t_2 = \epsilon\sqrt{(1 - 1/q^2)}$, then $f(y) = 1/q$. Hence the ray in A in the y direction is $[0, y/q]$. Now $|y| = \epsilon$ so y is not in A. But y is in $C + C$. Thus $C + C \not\subset A$.

In the general case $\dim E \geqq 2$, let e_1 and e_2 be linearly independent vectors in E. Then the subspace $F = \operatorname{sp}(e_1, e_2)$ spanned by e_1 and e_2 is isomorphic to E^2 under the map I such that $Iu_1 = e_1$, $Iu_2 = e_2$. If $B = I(A)$, B as a subspace of F has all the properties proved above for A. Let G be a subspace complementary to F. Then $G + A$ is radial at 0 in E and is circled.

There is no set C in E which is radial at 0 and such that $C + C \subset A + G$ (and thus the circled sets radial at 0 do not yield a vector topology). Suppose that there is such a C. Then $D = C \cap F$, considered as a subset of F, is radial at 0 and $D + D \subset A$. But this contradicts our result for E^2.

If dim $E \leqq 1$, the family of circled sets radial at 0 is a base for the Euclidean topology.

REMARK. In the real case for E^2 the proof is simpler and more intuitive if we let $f(x)$ be the angle between x and e_1.

References

1. N. Bourbaki, Espaces vectoriels topologiques, Actualités Sci. Ind., 1189 and 1229, Hermann, Paris, 1953 and 1955.

2. J. L. Kelley, Linear topological spaces, Van Nostrand, Princeton, N. J., 1963.

Reprinted from the AMERICAN MATHEMATICAL MONTHLY
Vol. 72, No. 5, May, 1965