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A FAVORABLE SIDE BET IN NEVADA BACCARAT*

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A winning strategy is developed for the nine to one *side bet* on a Banker natural nine. Let n be the number of cards that remain for play. Let t be the number of nines that remain for play. If $p(n, t)$ is the probability of a natural nine when n and t are given, then $p(n, t)$ is greater than 0.1 frequently enough to make the counting of n and t the basis for a practical winning strategy. The Kelly criterion (play to maximize the expected value of the log of capital) is used to determine bet sizes for favorable situations. Similar strategies are developed for the side bets on Banker natural eight, Player natural nine, and Player natural eight. The relationships between the four side bets are analyzed. It is shown that the *main* (roughly 1:1) bets on Banker and Player occasionally favor the player. There are theoretical favorable strategies but none exists which is currently practical.

1. INTRODUCTION

The games of Baccarat and Chemin de Fer are well known gambling games played for high stakes in several parts of the world. Baccarat is said to be a card game of Italian origin that was introduced into France about 1490 A.D. Two forms of the game developed. One form was called Baccarat and the other was called Chemin de Fer. The most basic difference between these two games is simply that three hands are dealt in Baccarat (called Baccarat en Banque in England) and two hands are dealt in Chemin de Fer (called Baccarat-Chemin de Fer in England and in Nevada). The cards Ace through nine are each worth their face value and the cards Ten, Jack, Queen and King are each worth zero points. A hand is evaluated as the sum modulo ten of its cards, i.e. only the last digit of the total is counted. The object of the game is to be as close to eight or nine as possible with two cards, or as close to nine as possible with at most three cards if one does not have eight or nine on his first two cards. Then the high hand wins.

The games of Baccarat and Chemin de Fer became popular in public casinos all over Europe, as well as in private games, about 1830. At the present time one or both of these games are well known in London, southern France, the Riviera, Germany and Nevada. A form of Chemin de Fer, which we shall call Nevada Baccarat, has been played in a few Nevada casinos since 1958.

The rules, structure and format of the three games have strong similarities. We studied Nevada Baccarat most intensively because the casinos where it

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was played were readily accessible. Our techniques can be carried over to the other forms of Baccarat and Chemin de Fer.

We were originally motivated by the observation that Baccarat and Chemin de Fer have several points of resemblance to the game of Blackjack, or Twenty-One. The fact that practical winning strategies for Twenty-One have been recently discovered suggested that there might also be practical winning strategies for Baccarat and Chemin de Fer [14], [15], [16]. In contrast to the situation in Twenty-One, we found that there are no current practical winning strategies for the main part of the game, i.e., for the money Banker and Player bets.*

During this work we learned that the Nevada version of the game sometimes had certain associated side bets. In the greater part of this paper we show that there are practical winning strategies for these side bets and we develop the details of a strategy based on the Kelly criterion.

Our favorable strategies were successfully exploited by players. This forced the casinos to remove the (once profitable) side bets from their layouts. We will talk in the present tense about these side bets even though, as of this writing, they are out of favor. There is a picture of the former layout at the Sands, complete with side bets [13, p. 419].

2. RULES AND PROCEDURES

We now describe the game of Nevada Baccarat. To begin the game, eight decks of cards are shuffled and a joker is placed face up near the end. The cards are then put into a wooden dealing box called a shoe. The first card is exposed, and its value is noted, face cards being counted as tens. Then this number of cards is discarded, or "burned."

The table has twelve seats, occupied by an assortment of customers and shills. A shill is a house employee who bets money and pretends to be a player in order to attract customers or stimulate play. We refer to them indiscriminately as "players." There are two principal bets, called "Banker" and "Players." Any player may make either of these bets before the beginning of any round of play, or "coup."

To begin the evening's play, two of the players are singled out. One is termed The Banker and the other is termed The Player. The seats are numbered *counter*clockwise from one to twelve. Player number one is initially The Banker, unless he refuses. In this case the opportunity passes counterclockwise around the table until someone accepts. The Player is generally chosen to be that player, other than The Banker, who has the largest bet on the Player. We have not noticed an occasion when there were no bets on The Player. When we played, there were shills in the game and they generally bet on The Player (except when acting as The Banker, when they generally bet on The Banker).

The Banker retains the shoe and deals as long as the bet "Banker" (which we also refer to as a bet on The Banker) does not lose. When the bet "Players" (which we also refer to as a bet on The Player) wins, the shoe moves to the player on the right. This player now becomes The Banker. If the coup is a tie, the players are allowed to alter their bets in any manner they wish. The same Banker then deals another coup.

To begin a coup, The Banker and The Player are dealt two cards each. As we noted above, the cards Ace through nine are each worth their face value and tens and face cards are each worth zero points. Only the last digit in the total is counted.

After The Banker and The Player each receive two cards, the croupier faces their hands. If either two-card total equals 8 or 9 (termed a natural 8 or a natural 9, as the case may be), all bets are settled at once.

If neither The Player nor The Banker have a natural, The Player and The Banker then draw or stand according to the set of rules in Table 1.

The high hand wins. If the hands are equal, there is a tie and no money changes hands. Players are then free to change their bets in any desired manner. If the coup being played is complete when the joker is reached, the shoe ends and the cards are reshuffled. Otherwise the coup is first played out to completion. Then the shoe ends and the cards are reshuffled. However, the casino may reshuffle the cards at any time between coups.

3. THE MAIN BETS

Two main bets against the house can be made. One can bet on either The Banker or The Player. Winning bets on The Player are paid at even money. Winning bets on The Banker are paid 0.95 of the amount bet. The five per cent tax which is imposed on what otherwise would have been an even-money payoff is called "vigorish." For eight complete decks, the probability that The Player wins is 0.446247, the probability that The Banker wins is 0.458597, and the probability of a tie is 0.095156.

The basic idea of the calculation of these numbers is to consider all possible distinct 6-card sequences. The outcome for each sequence is computed and the corresponding probability of that sequence is computed and accumulated in

TABLE 1. NEVADA BACCARAT RULES

Player having		
0–5	draws a card	
6–7	stands	
8–9	turns cards over	
Banker having	draws when The Player draws	does not draw when The Player draws
0	none, 0–9	
1	none, 0–9	
2	none, 0–9	
3	none, 0–7, 9	8
4	none, 2–7	0, 1, 8, 9
5	none, 4–7	0–3, 8, 9
6	6, 7	none, 0–5, 8, 9
7	stands	stands
8	turns cards over	turns cards over
9	turns cards over	turns cards over

* Some of the strategies which win theoretically, but currently seem impractical, may be practical when technology has advanced further. Further, we only rule out strategies based on card counting. Strategies based upon the analysis of card shuffling could conceivably yield practical winning strategies [17, Sec. 3, Ch. 4].

the appropriate register. Numerous short cuts, which simplify and abbreviate the calculation, are introduced. For example, only those sequences where the rules are not symmetric need to be considered. Also, sequences where, say, The Banker's (or The Player's) first two cards are $x, y (x \neq y)$ and y, x need not be considered separately.

The house advantage (we use advantage as a synonym for mathematical expectation) over The Player is 1.2351 per cent. The house advantage over The Banker is 0.458597×5 per cent $-$ 1.2351 per cent or 1.0579 per cent, where 2.2930 per cent is the effective house tax on The Banker's winnings. If ties are not counted as trials, then the figures for house advantage should be multiplied by 1/0.904844, which gives a house advantage per bet that is not a tie, over The Banker of 1.1692 per cent and over The Player of 1.3650 per cent. The effective house tax on The Banker in this situation is 2.5341 per cent.

Figures for the house advantage have been given by Scarne [13, p. 427], but seem, upon comparison with our correct figures, to be computed as though ties are discounted but the assertion is made that the figures apply to "... every hundred dealt hands in the long run ...". For reference, they are a 1.34 per cent advantage of The Banker over The Player (compares with our 1.2351 per cent and 1.3650 per cent figures), a 2.53 per cent charge on The Banker's winnings, (compares with our 2.2930 per cent and 2.5341 per cent), and a house advantage of 1.34 per cent over The Player, and 1.19 per cent over The Banker. Thus, *if* the confusion over ties is clarified, the figures given by Scarne are quite accurate for the case where ties are not included in counting the number of trials.

We attempted to determine whether or not the abnormal compositions of the shoe, which arise as successive coups are dealt, give rise to fluctuations in the expectations of The Banker and The Player bets which are sufficient to overcome the house edge. It turns out that this occasionally happens but the fluctuations are not large enough nor frequent enough to be the basis of a practical winning strategy. This was determined in two ways. First, we varied the quantity of cards of a single numerical value. The results were negative.

We next inquired as to whether, if one were able to analyze small n situations perfectly (e.g. the player might receive radioed instructions from a computer), there were appreciable player advantages on either bet a significant part of the time. We selected 29 sets of 13 cards each, each set drawn randomly from eight complete decks. There were small positive expectations in only two instances out of 58. Once The Player had a 3.2% edge and once The Banker had a 0.1% edge.

We next proved, by arguments too lengthy and intricate to give here, that the probability distributions describing the conditional expectations of The Banker and The Player spread out as the number n of unplayed cards decreases [17, Sec. 3, Ch. 2]. Thus there are fewer advantageous bets of each type, and they are less advantageous, as n increases above 13. The converse occurs as n decreases below 13.

The observed practical minimum n ranged from 8 to 17 in one casino and from 20 up in another. The theoretical minimum, when no cards are burned, is $n=6$. Thus the results for $n=13$ seem to conclusively demonstrate that no *practical* winning strategy is possible for the Nevada game, even with a computing machine playing a perfect game. Further considerations [17, Sec. 2, Ch. 2] show that a computing machine playing a perfect game might yield a practical winning strategy in those continental games where The Banker's strategy is completely optional. The situation is marginal.

4. THE SIDE BETS

The game of Nevada Baccarat sometimes has certain associated side bets. One can bet that The Banker has a natural nine. One can also bet that The Banker has a natural eight. These same bets can sometimes also be made with respect to The Player. The side bets each pay 9 to 1 (equivalently, 10 for 1). For eight complete decks, the probability of a natural nine is 0.09490, a house advantage of 5.10 per cent. For eight complete decks, the probability of a natural eight is 0.09453, a house advantage of 5.47 per cent. These values were computed from equations (1) and (13) respectively. For a bet equally divided between the two naturals, the house advantage is 5.29 per cent. The corresponding figures given by Scarne are erroneous [13, p. 427].

5. THE ADVANTAGEOUS SITUATIONS

Let n be the number of cards in the shoe that thus far have not been used in play. Let t denote the number of these n cards with the value nine. Assume that the $n-t$ other cards have been selected at random from the 384 non-nines which were originally in the complete eight decks. Let two cards d_1 and d_2 be drawn from the n cards. By considering cases, we find the probability that The Banker's first two dealt cards total nine. The side bet has positive expectation if and only if this probability exceeds 0.1.

(a) $d_1 = 9, d_2 = 0$

$$\text{Prob}(d_1 = 9) = t/n. \ \text{Prob}(d_2 = 0 \mid d_1 = 9) = (n - t)/(3(n - 1)).$$

If $d_1 = 9$ and $d_2 = 0$ then $\text{Prob}(d_1 + d_2 = 9) = 1$.

$$\text{Prob}(d_1 = 9 \text{ and } d_2 = 0 \text{ and } d_1 + d_2 = 9) = (t(n - t))/(3n\,(n - 1)).$$

(b) $d_1 = 0$ and $d_2 = 9$

$$\text{Prob}(d_1 = 0 \text{ and } d_2 = 9 \text{ and } d_1 + d_2 = 9) = (t(n - t))/(3n(n - 1))$$

by reasoning similar to that for (a).

(c) $d_1 \neq 9, d_2 \neq 9$

$$\text{Prob}(d_1 \neq 9) = (n - t)/n. \ \text{Prob}(d_2 \neq 9 \mid d_1 \neq 9) = (n - t - 1)/(n - 1)$$

If $d_1 \neq 9$ and $d_2 \neq 9$ then $\text{Prob}(d_1 + d_2 = 9) = (8(32)(32))/(384)(383))$.

$$\text{Prob}(d_1 \neq 9 \text{ and } d_2 \neq 9 \text{ and } d_1 + d_2 = 9)$$
$$= (8(32)(32)(n - t)(n - t - 1)/(384(383))n(n - 1)).$$

We combine the three cases above and find that the probability of obtaining a total of nine by drawing two cards is:

$$P_{n,t} = (2(n - t)(32n + 351t - 32))/(1149n(n - 1)). \qquad (1)$$

For N decks, the corresponding formula turns out to be

$$P_{n,t}^{N} = (2(n-t)(4Nn+44Nt-4N-t))/(3n(n-1)(48N-1)). \quad (2)$$

We find the result for the infinite deck case either by taking the limit of this expression as $N\to\infty$ and $r=n/t$ is held fixed, or by an easy direct calculation. The result is

$$P_r = (1-1/r)(1+11/r)/18. \quad (3)$$

Division shows that

$$P_{n,t}/P_r = 1 + ((n-t)(n-1) + (12)351t)/(383(n-1)(n+11t)). \quad (4)$$

Thus $P_{n,t} > P_r$. This is an important result for it shows that a strategy based on assuming an infinite deck will be conservative. Thus we can design a winning strategy based on the much simpler infinite deck calculations. The equality also shows that the approximation is "good." The corresponding result for N decks is

$$P_{n,t}^{N}/P_r = 1 + ((n-t)(n-1) + 12t(44N-1))/((48N-1)(n-1)(n+11t)). \quad (5)$$

We next determine how the favorable side bets are distributed. Given n cards, the probability that t of these cards are nines is given by

$$a_{n,t} = C(32;t)C(384;n-t)/C(416;n) \text{ where } C(r;s) \text{ is } r!/(s!(r-s)!). \quad (6)$$

This yields the recursion formulas

$$a_{n,t} = a_{n-1,t}(n(385-n+t))/((n-t)(417-n)) \quad (7)$$

$$a_{n,t} = a_{n,t-1}((n-t+1)(33-t))/(t(384-n+t)). \quad (8)$$

Let N' be the total number of cards which are not used during play, i.e., N' is the sum of the numbers of unseen burned cards and the cards left in the shoe when the house decides to shuffle. Then $416-N'$ cards are actually dealt. Thus the number n', of cards which as yet have not been used at some instant during play of a shoe, ranges from 416 to $N'+1$.

Either 4, 5 or 6 cards are used during a coup. Thus the number n of cards, which as yet have not been used just before the play of some coup begins in a shoe, if one of the numbers 416, 412 to 410, 408 to $(N'+4)$ and all these values can generally be attained. It is n which is significant for our purposes. We define $N=(N'+4)$, the least value which n can attain during the play of the shoe. Of course N generally varies from one run through the eight decks (also termed a shoe) to another.

We assume that the various values of n between 416 and N occur with equal probability. This is only an approximation to what actually occurs but it appears to introduce negligible errors in our calculations. With this assumption, the probability that the first card which is drawn in a given coup during the play of a shoe is drawn from n cards, is $1/(417-N)$. Therefore, the probability that the draw of the first card to a hand is from n cards, t of which are nines, is given to good approximation by:

$c_{n,t}=a_{n,t}/(417-N)$, where we assume that n ranges uniformly over all values from 416 down through N. Complete tables of $c_{n,t}$ were obtained from this relation, the recursion formulas for $a_{n,t}$, and a high speed computer. The values of $p_{n,t}$ for all possible n and t were likewise determined via the computer. It turns out that when N is fairly small, say 20 or 30, favorable side bets can be placed about 20 per cent of the time. The advantage on (mathematical expectation of) these side bets runs as high as 89 per cent when N is 20.

6. THE GAMBLING SYSTEM

Consider a coin toss game in which p, the probability of our winning, is greater than $\frac{1}{2}$. Suppose we have V_0 units of capital initially and V_t units before trial t. Since $p>\frac{1}{2}$, the expectation is positive so we might bet to maximize our expectation. But this means we should bet our entire current capital of V_t units at each trial. Consequently, with probability 1 we can expect to be ruined.

Alternately we might bet 1 unit at each trial. This generally makes the ruin probability small compared with many other conceivable betting schemes. However, one's capital grows slowly.

An interesting compromise is explored by Kelly [8]. He proposes (the Kelly criterion) to maximize the expectation of the logarithm of the player's capital. In his work, he assumes that money is infinitely divisible and that the player bets a fixed fraction f of his capital at each trial. It turns out that there is a fraction f^* with the desirable property that if two players compete, one using f and the other f^*, the probability that $V_t(f^*)>V_t(f)$ tends to 1 as t tends to ∞.

In practice, the system of Kelly must be modified to fit reality. Because integral bets only are allowed, one can generally only approximate the optimal f^*V_t called for by Kelly. Further, when V_t is small, f^*V_t may be much less than one unit and a reasonable approximation is no longer possible.

Now we develop the Kelly ideas for our specific problem. Let $p_k>1/10$, $1\le k\le K$, be the conditional probability that the natural nine will win, given an event k that is known to occur with probability c_k. When k occurs, suppose the player bets a fraction f_k of his capital. If t bets are made there are t_k bets of type k with w_k wins and l_k losses, for $1\le k\le K$ (there are no ties when betting on the side bets so $w_k+l_k=t_k$). Denote the player's capital after t bets by V_t, and his initial capital by V_0. Then, since the side bet pays nine to one,

$$V_t = V_0 \prod_{k=1}^{K} (1+9f_k)^{w_k}(1-f_k)^{l_k}.$$

Define the exponential rate of growth, G, by

$$G = \lim_{t\to\infty} ((\log_2 (V_t/V_0))/t). \quad (9)$$

Note that $1/G$ will be the mean number of bets required to double the initial capital V_0. The Kelly criterion is to maximize G. If we let $\partial G/\partial f_k=0$ and solve for f_k, we find that

$$f_k = (10p_k-1)/9. \quad (10)$$

Hence

$$G_{\max} = \sum_{k=1}^{K} c_k(p_k \log_2(10p_k) + (1 - p_k)\log_2(10(1 - p_k)/9)). \qquad (11)$$

7. THE WINNING STRATEGY

All the formulas needed to construct the strategy are now available. Given that n cards remain, t of which are nines, and a corresponding probability $p_{n,t}$ of winning, certain values of $p_{n,t}$ satisfy $p_{n,t} > 1/10$. This is a conditional probability, given an event that occurs with probability $c_{n,t}$. For those pairs (n, t) such that $p_{n,t} > 1/10$, the optimal fraction is

$$f_{n,t} = (10p_{n,t} - 1)/9. \qquad (12)$$

If $p_{n,t} \leq 1/10$, let $f_{n,t} = 0$, corresponding to no bet. Thus a table was computed telling the player for each possible pair (n, t) what fraction of his initial capital should be wagered. The table is too large to be practical for actual betting. However, equation (4) suggests, and examination of the computed values of $f_{n,t}$ verifies, that over nearly all of the range of interest, f_{n_1,t_1} is approximately equal to f_{n_2,t_2} if $n_1/t_1 = n_2/t_2$. Therefore the betting tables can be reduced to a very compact size. The use of this strategy in actual play will be described later.

If $N = 10$, the mean number of hours necessary to double initial capital (Banker natural nine bet only) is 45. If $N = 35$, the mean number of hours is 96. We will see that these figures can be divided by 2 and 4 respectively after the other side bets have been considered.

8. THE PROBLEM OF GAMBLER'S RUIN

Ideally, we would like to obtain a practical algorithm for computing the probability of ruin, starting with a given capital and a given set of casino rules, when betting The Banker natural nine in Nevada Baccarat. This appears to be an extremely complicated and lengthy problem. However, we will obtain upper and lower bounds for the ruin probabilities in a class of simpler situations.

Recall that $p_k > 1/10$, $1 \leq k \leq K$, was a conditional probability, given an event that occurs with probability c_k, and that

$$V_t = V_0 \prod_{k=1}^{K} (1 + 9f_k)^{w_k}(1 - f_k)^{l_k}.$$

Let us consider one of these situations for a fixed value of k. Since k is fixed, we will use the notation

$$V_t = V_0(1 + 9f)^w(1 - f)^l$$

where w is the number of wins and l is the number of losses, in t trials. The letter f represents the optimal fraction determined by the Kelly System. We will obtain upper and lower bounds for the ruin probabilities in such situations.

If bets of arbitrary size were allowed, the player would never be ruined,

provided he avoided ever betting his entire current capital. (In this case we would redefine ruin to mean that the player's capital tends to zero with probability one. Ruin is then possible once again.) In the more realistic situation where there is a minimum allowable bet m, and the player's capital becomes $V_t < m/f$, it will no longer be possible for the player to follow the Kelly strategy. In this case we say the player is ruined, although this differs slightly from the usual definition and from the above redefinition.

In what follows, we will assume that any size bet larger than m is allowable. This does not correspond to reality, because the casinos only allow bets of multiples of m, up to a certain maximum. The assumption enlarges the set of permissible strategies. It permits a better strategy "fit" than is obtainable in reality. This tends to increase the computed figure for the exponential rate of growth above its true value. We have shown that the errors introduced are small.

Ruin occurs when $V_t < m/f$, or taking $m = 1$, when $V_t < 1/f$. This ruin occurs when

$$V_0(1 + 9f)^w(1 - f)^t < 1/f$$

or when

$$w \log(1 + 9f) + l \log(1 - f) < -\log(V_0 f),$$

which we write as

$$wc_1 - lc_2 < c_3.$$

This is equivalent to absorption *to the left of* the barrier $c_3 = -\log_k V_0 f$, in a one-dimensional random walk in which the particle starts at the origin and at each trial either moves left $c_2 = \log_b(1 - f)$ units with probability q or moves right $c_1 = \log_k(1 + f)$ units with probability p. The formally different but similar problem where absorption occurs at the barrier, has a well-known solution in the event that c_1 and c_2 are integers,* and therefore in the event c_1 and c_2 are rationals.

We note the problems of absorption *at* the barrier and absorption *to the left of* the barrier are equivalent when c_2/c_1 is rational. For the particle's allowed positions x are a "lattice" of points on the real line, with a minimum spacing. (if $c_2/c_1 = a/b$ where a and b are integers and have no common factors, then the particle is limited to the set $x = kc_1/b$ where $k = 0, \pm 1, \pm 2, \cdots$.) If c_3 is not one of these points, absorption at c_3 and absorption to the left of c_3 mean the same thing. If c_3 is one of these points, consider the next such point x to the left. Then the problem of absorption to the left of c_3 is the same as the problem of absorption at $(x + c_3)/2$.

In general, c_1 and c_2 will not be rational. However, the problem reduces to the well-known case if we merely require that $r = c_2/c_1$ to be rational, for the inequality $c_1W_n - c_2L_n < c_3$ is equivalent to $W_n - rL_n < c_3/c_1$ and if $r = a/b$, where a and b are integers and we may assume a/b is in lowest terms, we have $bW_n - aL_n < b(c_3/c_1)$. Since our particle takes integral steps, this is equivalent to absorption at c, where c is the greatest integer less than $b(c_3/c_1)$.

* If c_3 is not an integer, absorption at c_3 is the same as absorption at $[c_3]$, the greatest integer less than c_3.

The condition $c_2/c_1 = a/b$ is equivalent to $-\log_b(1-f)/\log_b(1+f) = a/b$ or $(1+f)^a(1-f)^b = 1$ where $a > b > 0$. It is easy to show that there is a unique $f_0 > 0$ which satisfies this equation, and that f_0 increases from 0 to 1 as a/b increases from 1 to ∞. For example, when $a/b = 0.2/0.1 = 2$, we must find the positive root of $(1+f)^2(1-f) = 1$. It is $f_0 = (-1+\sqrt{5})/2 = 0.618$.

The case where $c_2/c_1 = a/b$ is useful for the construction of examples. However, the numbers c_1 and c_2 arise from f, which is given in advance, and they are usually such that c_2/c_1 is not rational. To solve this general case, we approximate by solutions of cases where c_2/c_1 is rational. If the particle takes steps of size 1 to the right with probability p and steps of size $r_i = a_i/b_i > c_2/c_1$ to the left with probability q, then clearly the probability U_i of absorption below c_2/c_1 is larger than the true value. Similarly, if $1 < s_i = e_i/f_i < c_2/c_1$, where e_i and f_i are positive integers, and the steps to the left are instead of this size, the probability L_i of absorption is smaller than the true value. Thus, if A is the probability of absorption in the original problem, $L_i \le A \le U_i$. We choose sequences $\{r_i\}$ and $\{s_i\}$ of rationals, such that $r_i \downarrow c_2/c_1$ and $s_i \uparrow c_2/c_1$. Then as i increases, $U_i \downarrow U \ge A$ and $L_i \uparrow L \le A$.

We observe that $L = A$. To see this, imagine the successive locations of our particle plotted as points (n, x) in the $N-X$ plane, where n is the number of trials and x is the corresponding location. Then each particular occurrence of the random walk is plotted as a polygonal line in the plane. Then, given $\epsilon > 0$ there is an $N(A, \epsilon)$ such that the probability of ruin after time $N(A, \epsilon)$ for the original random walk Q giving rise to A is less than ϵ. Now consider a random walk S_i which provides one of the L_i approximations to L. As i increases, the polygonal paths for a given S_i move down towards the corresponding polygonal paths for Q. There are only a finite number of distinct polygonal paths for Q and for a given S_i if we restrict consideration to $n \le N(A, \epsilon)$. Consider those vertices (if any) which lie in the absorbing region. Since the absorbing region is an open lower half-plane, there is a positive minimum distance d between these vertices and the boundary of the absorbing barrier. For all sufficiently large i, we can guarantee that when $n \le N(A, \epsilon)$ the paths for S_i are within $d/2$ of the corresponding paths for Q. Thus ruin under Q and under S_i always occurs "simultaneously" when $n \le N(A, \epsilon)$. Since both ruin probabilities are less than ϵ for $n > N(A, \epsilon)$, we see that $A - L_i < \epsilon$. Since ϵ is arbitrary $L_i \uparrow A$.

We observe that $U = A$ "usually" but that this is not always the case. We assume that c_2/c_1 is irrational, since the case of c_2/c_1 rational is solved, as remarked earlier. It follows from the fact that c_2/c_1 is irrational that the positions occupied by the particle, or equivalently, the vertices of the polygonal paths, get arbitrarily close to the barrier boundary $x = c_2/c_1$.

Case 1: The barrier boundary is not a point of occupancy for the particle. In this case $U = A$. To see this, consider the random walk R_1 giving rise to U_1. Given $\epsilon > 0$ choose $N(R_1, \epsilon)$ such that the probability of ruin under U_1 is less than ϵ when $n > N(R_1, \epsilon)$. From our hypothesis there is a minimum positive distance d to the barrier from those A-vertices lying above the barrier and having $n \le N(R_1, \epsilon)$. For i sufficiently large, we can ensure that the R_i-vertices are within $d/2$ of the corresponding A-vertices. Hence for such i, and $n \le N(R_1, \epsilon)$, ruin is "simultaneous" for R_i and Q. But for $n > N(R_1, \epsilon)$, the probability of ruin for Q is less than that for R_i which is less than that for R_1 which is less than ϵ. Hence $U_i - A < \epsilon$.

Case 2: The barrier boundary is a point of occupancy for the particle. In this case $U > A$. Under Q, the particle has a positive probability p of being on the barrier boundary. This is not ruin. However, the polygonal paths for U_i lie strictly below those for Q. Hence the corresponding paths for U_i lead to ruin. Thus $U - A \ge p > 0$. It is easy to see that $U - A = p > 0$.

The barrier boundary is a point of occupancy for the particle under Q if and only if there are integers m and n such that $mc_1 + nc_2 = c_3$. Hence this is equivalent to $U > A$.

We approximated the probability A of ruin in the associated random walk, by determining bounds for U_i and L_i and hence for A [5, Ch. 14, Sec. 8]. We computed extensive tables of these values. For example, if V_0 is 200 minimum units and f assumes the values .006, .036 and .093, then we have $.30 \le A \le .83$, $.04 \le A \le .14$ and $.01 \le A \le .05$ respectively. If V_0 is 1000 minimum units and f assumes the values .006, .036 and .093, then we have $.06 \le A \le .17$, $0 \le A \le .03$ and $0 \le A \le .01$ respectively.

9. THE EFFECTS OF APPROXIMATING THE OPTIMAL FIXED FRACTION

We earlier pointed out that in actual play only multiples of the minimum bet are allowed. Therefore it is in general not possible to bet precisely the quantity $V_t f^*$. It turns out that the losses in G due to these necessary approximations to $V_t f^*$ are not serious. Extensive computation shows that in the case of bets on Banker natural nine, when the fraction f of V_t actually bet satisfies $0.5f^* \le f \le 1.5f^*$, then G satisfies $0.75G_{\max} \le G$ over the range of interest $0 \le f^* \le 0.13$.

10. THE BANKER NATURAL EIGHT SIDE BET

A treatment, similar to that for The Banker natural nine, could be applied to The Banker natural eight side bet, yielding a strategy for this side bet. However, we will show that the strategy for The Banker natural eight is, for all practical purposes, identical to that for The Banker natural nine. The only difference is that one counts the number of eights rather than the number of nines.

The only formula changed is the one for $p_{n,t}$. To distinguish, the notations $\overset{8}{p}_{n,t}$ and $\overset{9}{p}_{n,t}$ will be used.

By considering cases as in section 5, we can obtain for The Banker natural eight bet a set of equations corresponding to those already obtained for The Banker natural nine.

The probability of obtaining a total of eight by drawing two cards (from eight well shuffled decks) is

$$\overset{8}{p}_{n,t} = ((n-t)(127n + 1405t - 127))/(2298n(n-1)). \tag{13}$$

It follows from (1) and (13) that

$$\left| \overset{9}{p}_{n,t} - \overset{8}{p}_{n,t} \right| \le 1/2298. \tag{14}$$

As in (10), we have $f^i_{n,t} = (10p^i_{n,t} - 1)/9$, where $i = 8, 9$. From this we see that if

$$\left| p^9_{n,t} - p^8_{n,t} \right| \le \epsilon \text{ then}$$

$$\left| f^9_{n,t} - f^8_{n,t} \right| \le 10\epsilon/9. \tag{15}$$

From (14) we have $\epsilon = 1/2298$ so (15) yields

$$\left| f^9_{n,t} - f^8_{n,t} \right| \le 10/20682 < 0.0005.$$

Thus betting tables for The Banker natural eight bet agree, to a high approximation, with betting tables for The Banker natural nine bet.

Some indication of the relation between the natural eight and natural nine bets should be given. Calculations show that, over the range of practical importance, if the total number n_o of cards and the number s_o of eights are held fixed, then the probability of a natural eight increases monotonely as the number t_o of nines is increased. However, the probability of a natural eight does not rise to the "normal" amount until n_o/t has decreased to somewhere in the neighborhood of 9.0.

Similarly, it turns out that if the deck is rich in eights, the probability of a natural nine is decreased, compared with the typical distribution for the non-nine cards.

If one is placing bets on both The Banker natural nine and The Banker natural eight, perhaps these facts should be considered. However, for a typical N, say $N = 40$, simultaneous bets on both natural nine and natural eight are warranted only about a quarter of the time that bets on at least one natural are called for. Further, in the case of simultaneous bets, the amounts bet on each are generally rather different, so that one of them might be thought of as dominant. Therefore we do not feel that increased complication of the strategy is justified.

11. THE PLAYER NATURAL NINE SIDE BET

Now consider the bet that the player has a total of nine on his first two cards. Again let $p_k > 1/10$, $1 \le k \le K$, be the conditional probability that The Player natural nine will win, given an event k that is known to occur with probability c_k. When k occurs, suppose the player bets a fraction f_k of his capital that The Player will receive a natural nine, and also bets a fraction f_k of his capital that The Banker will receive a natural nine. We consider both these wagers as one bet by the player. We show that approximately the same fraction should be used separately on each of the two bets when both are made simultaneously, as is used when only the single bet on The Banker's total is made.

If t bets are made, there are t_k bets of type k with w^p_k wins and l^p_k losses for The Player's hand, and w^b_k wins and l^b_k losses for The Banker's hand, where $1 \le k \le K$. Denote the player's capital after t bets by V_t, and his initial capital by V_0.

Let the number of times in t trials such that a bet of type k is made and both The Player and The Banker have a natural nine be $w_k(b, p)$, such that The Banker has a natural nine and The Player does not be $w_k(b, p')$, such that The Banker does not have a natural nine and The Player does be $w_k(b', p)$, and such that neither has a natural nine be $w_k(b', p')$. Let $p_k(b, p) = \lim_{t\to\infty} w_k(b, p)/t_k$ and define $p_k(b, p')$, $p_k(b', p)$ and $p_k(b', p')$ similarly. Let

$$p_k = \lim_{t\to\infty} (w^b_k/t_k) \quad \text{and} \quad q_k = \lim_{t\to\infty} (l^b_k/t_k).$$

These limits exist with probability l. Since the cards are assumed to have been shuffled randomly, by symmetry $p_k(b, p') = p_k(b', p)$ and

$$p_k = \lim_{t\to\infty} (w^p_k/t_k), \qquad q_k = \lim_{t\to\infty} (l^p_k/t_k).$$

It is intuitively clear that whether The Banker gets a natural nine is generally not independent of whether The Player does. However, for simplicity, we assume independence in the next derivation.

Then we have

$$V_t = V_0 \prod_{k=1}^{K} \left((1 + 18f_k)^{(w^p_k w^b_k)/t_k}(1 + 8f_k)^{(w^p_k l^b_k)/t_k}(1 + 8f_k)^{(l^p_k w^b_k)/t_k}(1 - 2f_k)^{(l^p_k l^b_k)/t_k}\right).$$

Since

$$G = \lim_{t\to\infty} ((\log_2 (V_t/V_0))/t),$$

we have

$$G = \sum_{k=1}^{K} c_k(p_k^2 \log_2 (1 + 18f_k) + 2p_k q_k \log_2 (1 + 8f_k) + q_k^2 \log_2 (1 - 2f_k)).$$

If we set $\partial G^2 f_k = 0$ and $q_k = 1 - p_k$, we have

$$144f_k^2 + 2(50p_k^2 - 90p_k + 13)f_k - (10p_k - 1) = 0$$

By applying the quadratic formula and selecting the positive root we obtain

$$f_k^{*bp} = (1/144)(-(50p_k^2 - 90p_k + 13) + 5(100p_k^4 - 360p_k^3 + 376p_k^2 - 36p_k + 1)^{1/2}).$$

Recall that for The Banker natural nine only, the optimal fraction is

$$f^{*b} = (10p_k - 1)/9.$$

Calculations show that

$$f^{*bp} \le f^{*b} \quad \text{for } 0 \le f^{*b} \le 0.13. \text{ Also}$$

$$(f^{*b} - f^{*bp})/f^{*b} \le .002 \quad \text{for} \quad 0 \le f^{*b} \le 0.07,$$

$$(f^{*b} - f^{*bp})/f^{*b} \le .005 \quad \text{for } 0.07 \le f^{*b} \le 0.13.$$

Thus, the f^{*bp} obtained by assuming independence of the events "banker natural nine" and "player natural nine" agree to a good approximation, over the range of practical interest, with the values of f^{*b}. The question remains as to whether the assumption of independence gives f^{*bp} values which closely approximate the true values of f^{*bp}. Tedious computations, which we omit here, show that they in fact do over the range of practical interest.

Thus, the joint bet may safely be taken, for practical purposes, to be twice the bet on Banker natural nine only.

12. APPLICATION OF THE WINNING STRATEGY

As a result of the last section, our final strategy is to count total cards n, the number t of nines, and the number s of eights. Compute the ratio n/t and use the fraction corresponding to n/t to bet fV (where V is current capital) on natural nines. Compute the ratio n/s and use the fraction f corresponding to n/s to bet fV on natural eights.

In practice, one makes a table showing dollar amounts corresponding to several values of V. Then the numbers for that V value closest to the current one are memorized prior to each shoe.

A computer program was written to simulate the play of Nevada Baccarat. It verified that the system was valid, and that a casino try was justified.

The system was successfully tested in the casinos. We studied the side bets in two particular casinos, which we refer to as casinos A and B. In casino A, the limits on the main bet were \$5 to \$2000. Bets on the naturals ranged from \$5 to \$200 on natural eight and an additional \$5 to \$200 on natural nine. The bet on either natural could be divided between The Player and The Banker in desired proportions. In casino B, the limits on the main bet were \$20 to \$2000 and bets on The Banker naturals only were allowed. The limits were \$20 to \$200 on each of The Banker naturals.

An initial capital of \$4000 was used at casino A and initial capital of \$20,000 was used at casino B. We bet according to Table 2.

Table 2 has a column for the infinite deck case to show how that approximation compares with the exact calculations. To obtain this column, we use (3) and (10) and we find

$$f^* = (10p_r - 1)/9 = (-55/r^2 + 50/r - 4)/81.$$

The roots of $f^*=0$ are approximately $r=1.22$ and $r=11.28$ so $f^*>0$ if and only if $1.22<r<11.28$. Note that $df^*/dr=c(11-5r)/r^3$, where $c>0$. Thus $f^*(r)$ increases from 0 to a maximum at $r=2.2$ and then decreases to 0, as r increases from 1.22 to 11.28. As we would expect from (4), and from comparing the entries in Table 2 under the ∞ column with those under the other columns, the values under those other columns in Table 2 agree well with this. Examination of the results shows that the original table can be recovered from this abbreviated version, via interpolation and extrapolation, with errors generally of the order of 0.001. A square in the table corresponds to a possible situation if and only if $t \leq n$ and $t \leq 32$. The column labled "∞" gives the value of f^* for the infinite deck approximation.

We are indebted to the referees for several helpful suggestions.

TABLE 2.* BETTING FRACTION f^* OF CAPITAL FOR $1.3 \leq n/t \leq 11.6$

	n					
n/t	20	40	80	130	≥240	∞
1.3	029	026				024
1.4	052	048				045
1.5	068	064				060
1.6	079	075				071
1.7	087	083				079
1.8	092	088				084
1.9	096	091				087
2.0	098	093				090
2.1	099	094				091
2.2	099	095				091
2.5	097	093	091			089
3.0	089	084	082			081
4.0	069	065	064			063
5.0	052	049	048	047		047
6.0	039	037	035	035		035
7.0	029	027	026	025		025
8.0	020	019	018	017	017	017
9.0	013	012	011	011	011	011
10.0	008	007	006	006	006	006
10.8	004	003	002	002	002	002
11.6	000	000	000	000	000	000

* Table entries are understood to be preceded by a decimal. This table was originally computed for all possible values of n and t.

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